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A UNIFIED APPROACH OF SOME CHARACTERIZATION RESULTS

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Abstract- This paper considers the problem of identifying some probability distributions using the concepts of right and left truncated moments of some function of the k^{th} order statistic. Furthermore, it investigates the necessary and sufficient conditions to characterize distributions by the equation $E(h(Y_k)|Y_k \prec t) = \phi_k(t)$ as well as $E(h(Y_k)|Y_k \succ t) = \theta_k(t)$ Some wellknown results follow from our results as special cases.

Keywords- Characterization, truncated moments, exponentiated Weibull, inverse Weibull, exponentiated Pareto, Burr, Power, uniform and Ferguson distributions

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Introduction

Characterization theory is a very interesting branch of science that locates on the borderline between mathematics and statistics. It utilizes many facts and tools of mathematical analysis such as measure Theory, differential equations, complex variable, and integral equations. It deals with the characteristic properties of probability distributions so that it helpes researchers to determine distributions uniquely. Some excellent references are, e.g., Azlarov and Volodin [5], Kagan, Linnik and Rao [15], Mchlachlan and Peel [18], among others. A large number of ideas and concepts have been used to identify the probability distributions. Gupta [13], Ouyang [22], Talwalker [25] and Elbatal, et al. [8] have used the concept of right truncated moments to characterize different distributions like Weibull, gamma, beta, exponential, Burr, Pareto and Power distributions. Really, characterizations via right truncated moments are very important in practice since, in some situations, some measuring devices may be unable to record values greater than time t. On the other hand there are some measuring devices that can not be able to record values smaller than time t. This has encouraged several authors and researchers to deal with the problem of identifying distributions by means of left truncated moments, see, e.g., Glänzel,, et al. [11], Gupta [13], Ahmed [2], Navaro et.al. [20] and Fakhry [9]. Furthermore, several authers have used the concept of order Statistics to characterize different probability distributions, see, e.g., Obretenov [21], Khan and Beg [16], Fakhry [9], Govindarajulu [12] and A-Rahman [4].

Main Results.

Let X be an absolutely continuous random variable with distribution function $F(\cdot)$ defined on (α, β) Denote by $y_1 \prec y_2 \prec \cdots \prec y_n$ the order statistics associated with a random sample of size *n* from $F(\cdot)$

Let $g(\cdot)$ and $H(\cdot)$ be two differentiable functions defined on (α, β) Elbatal, et al. [8] have characterized some types of probability distributions through the equation $E(g(X)|X \prec y)) = g(y) - H(y).$ On the other hand, Dimaki and Xekalaki [7] have identified some probability distributions using the equation $E(g(X)|X \succ y) = g(y) + H(y).$ Furthermore, Su and Huang [24] have discussed the necessary and sufficient conditions under which $E(g(X)|X \succ y) = \phi(y)$ characterizes some probability distributions.

In this paper, we generalize the above results using the concept of order statistics. Furthermore, we study the necessary and sufficient conditions to identify the probability distributions through the equation $E(g(Y_k)|Y_k \prec t) = \phi_k(t).$ The following Theorem generalizes the result of Elbatal, et al. [8].

Theorem (A)

Let X be an absolutely continuous random variable with density function $f(\cdot)$ and cdf $F(\cdot)$ such that $\lim_{x \to \beta^-} F(x) = 1$ and $F(\cdot)$ has continuous non-vanishing first order derivative on (α,β) so that in particular 0≤F (x)≤1 for all $x \in (\alpha, \beta).$ Let $Y_1 \prec Y_2 \prec \cdots \prec Y_n$ denote the order statistics of a random sample of size *n* from $F(\cdot)$. Assume that $h(\cdot)$ and $H_k(\cdot)$ are two non-vanishing real valued functions defined on (α,β) that have non-vanishing first order derivatives such that:

(a) $E(h(X)) \prec \infty$ **(b)** $\lim_{x \to \alpha^+} h(x) F^k(x) = 0$

(c) $\int_\alpha^\beta \frac{h'(y)}{kH_k(y)} = \infty$ Then

$$F(t)=\left(\frac{H_k(\beta)}{H_k(t)}\right)^{\frac{1}{k}} \exp-\int_t^{\beta}\frac{h'(y)}{kH_k(y)}dy \tag{1}$$

where $H_k(t)=\dfrac{\int_a^t h'(y)F^k(y)dy}{F^k(t)}$

Iff $E(h(Y_k)|Y_k \prec t)=h(t)-H_k(t)$ (2)

Proof Necessity

The conditional density function of the *k*th order statistic $Y_k|Y_k \prec t$ [6] is given by:

$$f_n(y)=\frac{kf(y)[F(y)]^{k-1}}{F^k(t)},\quad y\in(\alpha,t) \tag{3}$$

Therefore, by definition we have:

$$E(h(Y_k)\mid Y_k \prec t)=\frac{\int_\alpha^t h(y)dF^k(y)}{F^k(t)} \tag{4}$$

Integrating by parts and making use of the assumption that $\lim_{x\to\alpha^+} h(x)F^k(x)=0,$ one gets:

$$E(h(Y_k)|Y_k \prec t)=h(t)-\frac{\int_\alpha^t h^{\backslash}(y)F^k(y)dy}{F^k(t)}=h(t)-H_k(t)$$

Sufficiency

[Eq-2] can be written as an equation of the unknown *F*(·) as follows:

$$k\int_\alpha^t h(y)f(y)F^{k-1}(y)dy=[h(t)-H_k(t)]F^k(t)$$

Differentiating both sides with respect to t, cancelling out the term $kh(t)f(t)[F(t)]^{k-1}$ from both sides, adding to both sides $kH_k(t)f(t)[F(t)]^{k-1}$ and dividing the result by $kH_k(t)[F(t)]^k$ one gets:

$$\frac{f(t)}{F(t)}=\frac{h'(t)-H_k^{\backslash}(t)}{kH_k(t)}$$

Integrating both sides with respect to t from *x* to β and using the fact that $\lim_{x\to\beta^-}F(x)=1$ then, after some elementary computation one get:

$$F(x)=\left[\frac{H_k(\beta)}{H_k(x)}\right]^{\frac{1}{k}}\exp-\int_x^\beta\frac{h^{\backslash}(y)}{kH_k(y)}dy$$

Remarks (A)

1. If we set *k*=1 in Theorem (A), we obtain a characterization in terms of the minimum. Furthermore, using [Eq-3] and [Eq-4] we have : $E(h(Y_1)|Y_1 \prec t)=E(h(X)|X \prec t).$

Therefore, we can say that Theorem (A) generalizes Theorem (C) obtained by Elbatal, et al [8].

2. If we put *k*=*n* in Theorem (A), we obtain a characterization in terms of the maximum.
3. If we put n= 2r + 1 and k= r + 1in Theorem (A), we obtain a characterization in terms of the median.
4. If we put $h(X)=X,\ H_k(X)=[kc+1]^{-1}X,\ \alpha=0$ and $\beta=1$ in Theorem (A), where *c* is a positive parameter, we obtain a characterization concerning the power distribution with parameter *c*. For *c*=1 we have the uniform distribution.
5. If we set $h(X)=-X^{-b},\ H_k(X)=\frac{c}{k},\ \alpha=0$ *and* $\beta=\infty$ in Theorem (A), where *b* and *c* are positive parameters, we obtain a characterization concerning the inverse Weibull distribution.
6. If we set $h(X)=-\exp-cX^b,\ H_k(X)=[k\theta+1]^{-1}[1-\exp-cX^b],\ \alpha=0,\ \beta=\infty$ in Theorem (A), we obtain a characterization concerning the exponentiated Weibull distribution with positive parameters $\theta,\frac{1}{c},$ and b [19]. For θ=1, we have a characterization for the Weibull distribution with parameters $\frac{1}{c}$ and b. For θ=1 and b=1, we obtain the exponential distribution.
7. If we set $h(X)=-\left(\frac{b}{X}\right)^a,\ H_k(X)=\frac{1-\left(\frac{b}{X}\right)^a}{k\theta+1},\ \alpha=b,\ \beta=\infty$ in Theorem (A), where *a, b* and θ are positive parameters, we obtain a characterization concerning the exponentiated Pareto distribution of the first type with parameters a, b and θ [17]. For θ= 1, we have the Pareto distribution of the first type.
8. If we set $h(X)=-(1+X)^{-a},H_k(X)=\frac{1-(1+X)^{-a}}{k\theta+1},\ \alpha=0,\ \beta=\infty$ in Theorem (A), where α and θ are positive parameters, we have a characterization concerning the exponentaited Pareto of the second type [1]. For θ= 1, we have the Pareto distribution of the second type.
9. If we set $h(X)=[n-g(X)]$ and $H_k(X)=(kc+1)^{-1}[n-g(X)]$ in Theorem (A), where c is a parameter such that: c∉ {-1,0} and g(x) is a differentiable function defined on (α, β) such that:
(a) $g(x)\neq n\forall x\in(\alpha,\beta)$ (b) $\lim_{x\to\alpha^-}g(x)=n$ (c) $\lim_{x\to\beta^-}g(x)=n-1$ We have a characterization for a random variable *X* with cdf defined by $F(x)=[n-g(x)]^c$ [22].
10. If we set $h(X)=X,H_k(X)=(k\theta+1)^{-1}[X-\alpha]$ in Theorem (A), where θ, β and α are positive parameters such that $\alpha\prec x\prec\beta$ we have a characterization for a random variable *X* with cdf defined by $F(x)=\left[\frac{x-\alpha}{\beta-\alpha}\right]^\theta$ [10].
11. If we set $h(X)=X,H_k(X)=(k\theta-1)^{-1}[r-x],$ in Theorem (A), where *θ, r* and *d* are positive parameters such that $x\prec d,$ we have a characterization for a random variable X with cdf defined by $F(x)=\left[\frac{r-d}{r-x}\right]^\theta$ [10].
12. If we set $h(X)=X,H_k(X)=\frac{\theta}{k}$ in Theorem (A), where *d* and *θ* are positive parameters such that $x\prec d,$ we have a characterization for a random variable *X* with cdf defined by $F(x)=\exp\frac{x-d}{\theta}$ [10].

The following Theorem investigates the necessary and sufficient conditions that must be satisfied such that: $E(g(Y_k)|Y_k\prec t)=\phi_k(t)$ identifies some probability distributions.

Theorem (B)

Let *X* be an absolutely continuous random variable with density function f(·) and cdf F(·) such that $\lim_{x\to\beta^-}F(x)=1$ and F(·) has continuous non-vanishing first order derivative on (α, β) so that 0≤F(x) ≤1 for all x. Let $Y_1\prec Y_2\prec\cdots\prec Y_n$ denote the order statistics of a random sample of size *n* from F(·). Assume that (for every natural number *k* ≤ *n*), $\phi_k(\cdot)$ and $g(\cdot)$ are two continuous non-vanishing real valued functions defined on (α, β) such that: (a) $g(x)$ is a differentiable function. (b) $E(g(X)) \prec \infty$ (c) $\phi_k(x) - g(x) \neq 0 \quad \forall x \in (\alpha, \beta)$ Then

$$E(g(Y_k)|Y_k \prec t) = \phi_k(t), \qquad t \in (\alpha, \beta) \tag{5}$$

Iff

1. $\phi_k(\cdot)$ is a differentiable function.
2. $F(t) = \exp \int_t^\beta \left\{ \frac{\phi_k^\backslash(x)}{k[\phi_k(x) - g(x)]} \right\} dx = \left[\frac{\phi_k(\beta) - g(\beta)}{\phi_k(t) - g(t)} \right]^{\frac{1}{k}} \exp \int_t^\beta \left\{ \frac{g^\backslash(x)}{k[\phi_k(x) - g(x)]} \right\} dx$
3. $\lim_{t \to \alpha^+} \int_t^\beta \left\{ \frac{\phi_k^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = -\infty$
4. $\lim_{t \to \alpha^+} \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = -\infty$
5. The function $Q_k(t) = \int_t^\beta \left\{ \frac{\phi_k^\backslash(x)}{k[\phi_k(x) - g(x)]} \right\} dx$ is a non-positive increasing function.
6. $\lim_{t \to \alpha+} \phi_k(t) \exp \int_t^\beta \left\{ \frac{\phi_k^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = 0$

Proof Necessity

[Eq-5] is equivalent to the following equation:

$$k \int_\alpha^t g(x) f(x) [F(x)]^{k-1} dx = \phi_k(t) [F(t)]^k \tag{6}$$

To prove (1), we note that the continuity of the functions $f(x)$, $F(x)$ and $g(x)$ implies the continuity of their product. Therefore, $W(x) = f(x) g(x) [F(x)]^{k-1}$ function is integrable. Hence the left hand side of [Eq-6] is also differentiable. This means that the right hand side of [Eq-6] is also differentiable. But the function *F*(*x*) is differentiable. Therefore, the function $\phi_k(x)$ must be differentiable.

To prove (2), differentiate [Eq-6] with respect to t and dividing both sides by $[F(t)]^{k-1}$ one gets: $kg(t) f(t) = k\phi_k(t) f(t) + \phi_k^\backslash(t) F(t)$

Adding to both sides the term $-kg(t) f(t) - \phi_k^\backslash(t) F(t)$ and dividing the result by $k[\phi_k(t) - g(t)] F(t)$

we get: $\frac{f(t)}{F(t)} = \frac{-\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]}$

Integrating both sides with respect to t from *x* to *β*, noting that $\lim_{x \to \beta^-} F(x) = 1$ and performing some elementary computation, one gets:

$$F(x) = \exp \int_x^\beta \left\{ \frac{\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]} \right\} dt = \left[\frac{\phi_k(\beta) - g(\beta)}{\phi_k(x) - g(x)} \right]^{1/k} \exp \int_x^\beta \left\{ \frac{g'(t)}{k[\phi_k(t) - g(t)]} \right\} dt$$

Proof of (3)

The virtue that *F*(*x*) is a cdf implies that $\lim_{x \to \alpha^+} F(x) = 0$ Therefore, taking the limit of the last equation as $x \to \alpha^+$ and making use of the continuity of the exponential function, we get:

$$\lim_{x \to \alpha^+} \exp \int_x^\beta \frac{\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]} dt = \exp \lim_{x \to \alpha^+} \int_x^\beta \frac{\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]} dt = 0$$

Hence, $\lim_{x \to \alpha^+} \int_x^\beta \left\{ \frac{\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]} \right\} dt = -\infty$

Since *k* is a positive finite number, one gets:

$$\lim_{x \to \alpha} \int_x^\beta \left\{ \frac{\phi_k^\backslash(t)}{\phi_k(t) - g(t)} \right\} dt = -\infty$$

Proof of (4)

It is easy to see that (using condition 2 and the assumptions imposed on the functions $\phi_k(\cdot)$ and $g(\cdot)$

$$\lim_{t \to \alpha^+} [\phi_k(t) - g(t)] F^k(t) = \lim_{t \to \alpha^+} [\phi_k(\beta) - g(\beta)] \exp \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx =$$

$$[\phi_k(\beta) - g(\beta)] \lim_{t \to \alpha^+} \exp \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = 0$$

Therefore (since the exponential function is continuous), we get:

$$\lim_{t \to \alpha^+} \exp \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = \exp \lim_{t \to \alpha^+} \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = 0$$

This implies that: $\lim_{t \to \alpha^+} \int_t^\beta \left\{ \frac{g^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = -\infty$

Proof of (5)

The virtue that *F*(*x*) is a cdf implies that (a) $0 \le F(x) \le 1$ (b) *F*(*x*) is an increasing function. Therefore, using condition 2, we have: $Q_k(x) = \ln F(x) \le 0$ for all $x \in (\alpha, \beta)$. Moreover, differentiating the last equation with respect to *x*, we get: $Q_k^\backslash(x) = \frac{f(x)}{F(x)} \ge 0$ for all $x \in (\alpha, \beta)$ Hence $Q_k(x)$ is a non-positive increasing function.

Proof of (6)

[Eq-6] implies that $\lim_{t \to \alpha^+} \phi_k(t) F^k(t) = 0$ Therefore (using condition 2), we get: $\lim_{t \to \alpha^+} \phi_k(t) \exp \int_t^\beta \left\{ \frac{\phi_k^\backslash(x)}{\phi_k(x) - g(x)} \right\} dx = 0$

Sufficiency

At first, we note that the conditions 1, 3, 4 and 5 implies that the function *F*(*x*) defined by condition 2 is a cdf of some random variable *X*. Thus differentiating *F*(*x*) with respect to x one gets:

$$f(x) = \frac{-\phi_k^\backslash(x)}{k[\phi_k(x) - g(x)]} \exp \int_x^\beta \left\{ \frac{\phi_k^\backslash(t)}{k[\phi_k(t) - g(t)]} \right\} dt = \frac{-\phi_k^\backslash(x)}{k[\phi_k(x) - g(x)]} F(x)$$

Multiplying the last equation by $k[\phi_k(x) - g(x)][F(x)]^{k-1}$ and performing some elementary computation, one gets:

$$kg(x) f(x) [F(x)]^{k-1} = k\phi_k(x) f(x) [F(x)]^{k-1} + \phi_k^\backslash(x) [F(x)]^k = [\phi_k(x) F^k(x)]^\backslash$$

Integrating both sides of the last equation with respect to *x* from α to t, recalling that $\lim_{x \to \alpha+} \phi_k(x) F^k(x) = 0$ and dividing both sides by $F^k(t)$ one gets: $E(g(Y_k)|Y_k \prec t) = \phi_k(t)$

Example (A): Let X be an absolutely continuous random variable with cdf $F(\cdot)$. Let $Y_1 \prec Y_2 \prec \cdots \prec Y_n$ be the order statistics associated with a random sample of size n from $F(\cdot)$. Then, we can show that *X* has the uniform distribution with parameters α and β iff for any natural number *k* where 1≤*k*≤*n*, the following equation is satisfied:

$$E(Y_k|Y_k \prec t) = \frac{kt + \alpha}{k + 1}, \quad t \in (\alpha, \beta)$$

It is easy to see that both of functions $\phi_k(t) = \frac{kt + \alpha}{k + 1}$ and $g(t) = t$ satisfies the six conditions stated in Theorem (B).

Now, we study the dual case of Theorem (A) and generalize the result of Diemaki and Xekalaki [7].

Theorem (C)

Let X be an absolutely continuous random variable with density function f(·) cdf F(·) and survival function G(·) such that $\lim_{x\to\alpha^+} G(x)=1$ and F(·) has continuous non-vanishing first order derivative on (α, β) so that 0≤$F(x)$≤1 for all x. Let $Y_1 \prec Y_2 \prec \cdots \prec Y_n$ denote the order statistics of a random sample of size n from F(·) Assume that $h(\cdot)$ and $H_k(\cdot)$ are two non-vanishing differentiable functions on (α, β) such that:

(a) $E(h(X)) \prec \infty$ (b) $\lim_{x\to\beta^-} h(x)[G(x)]^{n-k+1}=0$ (c) $\int_\alpha^\beta \left\{\frac{h^{\backslash}(x)}{(n-k+1)H_k(x)}\right\}dx=\infty$

Then

$$G(t)=\left[\frac{H_k(\alpha)}{H_k(t)}\right]^{\frac{1}{n-k+1}}\exp-\int_\alpha^t\left\{\frac{h^{\backslash}(y)}{(n-k+1)H_k(y)}\right\}dy \qquad (7)$$

for any $t\in(\alpha,\beta)$

Where $H_k(t)=\frac{\int_t^\beta h^{\backslash}(y)[G(y)]^{n-k+1}dy}{[G(t)]^{n-k+1}}$

Iff

$$E(h(Y_k)|Y_k \succ t)=h(t)+H_k(t), \qquad t\in(\alpha,\beta) \qquad (8)$$

Proof. Necessity

The density function of the k^{th} order statistic $Y_k|Y_k \succ t$ [6] is given by:

$$f_n(y)=\frac{(n-k+1)f(y)[G(y)]^{n-k}}{[G(t)]^{n-k+1}} \qquad y\in(t,\beta) \qquad (9)$$

Therefore, by definition we have:

$$E(h(Y_k)|Y_k \succ t)=\frac{\int_t^\beta h(y)d[G(y)]^{n-k+1}}{[G(t)]^{n-k+1}} \qquad (10)$$

Integrating by parts and making use of the assumption that $\lim_{x\to\beta^-} h(x)[G(x)]^{n-k+1}=0$ one gets:

$$E(h(Y_k)|Y_k \succ t)=h(t)+\frac{\int_t^\beta h^{\backslash}(y)[G(y)]^{n-k+1}dy}{[G(t)]^{n-k+1}}=h(t)+H_k(t)$$

Sufficiency

[Eq-8] can be written in integral form as follows:

$$(n-k+1)\int_t^\beta h(y)f(y)[G(y)]^{n-k}dy=[h(t)+H_k(t)][G(t)]^{n-k+1}$$

Integrating by parts, recalling that $\lim_{x\to\beta^-} h(x)[G(x)]^{n-k+1}=0$ and cancelling out the term $h(t)[G(t)]^{n-k+1}$ from both sides, one gets:

$$\int_t^\beta h^{\backslash}(y)[G(y)]^{n-k+1}dy=H_k(t)[G(t)]^{n-k+1} \qquad (11)$$

Or equivalently $H_k(t)=\frac{\int_t^\beta h^{\backslash}(y)[G(y)]^{n-k+1}dy}{[G(t)]^{n-k+1}}$

Now, differentiate [Eq-11] with respect to t, adding to both sides $-H_k^{\backslash}(t)[G(t)]^{n-k+1}$ and dividing the result by $-(n-k+1)H_k(t)[G(t)]^{n-k+1}$ one gets: $\frac{f(t)}{G(t)}=\frac{h^{\backslash}(t)+H_k^{\backslash}(t)}{(n-k+1)H_k(t)}$

Integrating both sides with respect to t from α to x, recalling that $\lim_{x\to\alpha^+} G(x)=1$ and performing some elementary computation, one gets:

$$G(t)=\left[\frac{H_k(\alpha)}{H_k(t)}\right]^{\frac{1}{n-k+1}}\exp-\int_\alpha^t\left\{\frac{h^{\backslash}(y)}{(n-k+1)H_k(y)}\right\}dy$$

This completes the proof.

Remarks (C)

1. If we set k= n in Theorem (C), we obtain a characterization in terms of the maximum. Moreover, using [Eq-9] and [Eq-10], we have: $E(h(Y_n)|Y_n \succ t)=E(h(X)|X \succ t)$

   Therefore, we can say that Theorem (C) generalizes Theorem (A) obtained by Dimaki and Xekalaki [7].
2. If we set k =1 in Theorem (C), we obtain a characterization in terms of the minimum.
3. If we set n =2r+1 and k = r+1 in Theorem (C), we obtain a characterization in terms of the median.
4. If we put $h(X)=X^b, b\succ 0$ and $H_k(X)=c[n-k+1]^{-1}$ in Theorem (C), where c and b are positive constant, α =0 and β =∞, we obtain a characterization concerning the Weibull distribution with parameters c and b. For k =n, the result reduces to that of Hemdan[14]. For k =n and b =1, the result reduces to that of Shanbhag [23].
5. If we set $h(X)=-\ln(1-X^a), a\succ 0,\ H_k(X)=(n-k+1)^{-1},$ α= 0 and β= 1 in Theorem (C) we obtain a characterization concerning the Power distribution with positive parameter a. For a =1, we have a characterization for the uniform distribution. For k =n and a =1, the result reduces to that of Hemdan [14] characterizing the uniform distribution.
6. If we set $h(X)=\ln(1+X^a),\ H_k(X)=[c(n-k+1)]^{-1}$ in Theorem (C), where c and a are positive parameters, α= 0 and β= ∞, we obtain a characterization concerning Burr distribution. For a =1, we have a characterization concerning the Pareto distribution of the 2nd type with parameter c.
7. If we set $h(X)=X,\ H_k(X)=X[c(n-k+1)-1]^{-1}, \alpha=\theta\succ 0$ and $\beta=\infty$ in Theorem (C), we obtain a characterization concerning the Pareto distribution of the first type with parameter θ, c.
8. If we set $h(X)=(n+g(X)),\ H_k(X)=-[(n-k+1)c+1]^{-1}(n+g(X))$ in Theorem (C), where c is a parameter such that $c\notin\{-1,0\}$ and $g(x)$ is a differentiable function defined on (α, β) such that

   (a) $g(x)\neq -n \forall x\in(\alpha,\beta)$ **(b)** $\lim_{x\to\alpha^+} g(x)=1-n$ **(c)** $\lim_{x\to\beta^-} g(x)=-n$

   we have a characterization for a random variable X with survival function G(·) defined by $G(x)=[n+g(x)]^c$ [22].
9. If we put $h(X)=g(X)$ and $H_k(X)=\frac{c}{n-k+1}$ in Theorem (C) where c is a positive parameter and $g(x)$ is a differentiable function defined on (α, β), we obtain a characterization concerning the exponential family with parameter c [4].
10. If we set $h(t)=t^\alpha$ in Theorem (C), we obtain the result of Ahsanullah [3].

Now, we discuss the necessary and sufficient conditions to identify distributions by the equation $E(h(Y_k)|Y_k \succ t)=\theta_k(t)$

Theorem (D)

Let X be an absolutely continuous random variable with density function f(·), cdf F(·) and survival function G(·) defined on (α, β) with $F^{\backslash}(x) \succ 0$ for all x so that in particular 0≤F(x)≤1. Denote by $Y_1 \prec Y_2 \prec \cdots \prec Y_n$ the order statistics associated with a random sample of size n from F(·). Let g(·) and $\theta_k(\cdot)$ be two continuous non-vanishing real valued functions defined on (α, β) such that:

(a) $g(x)$ is a differentiable function. (b) $E(g(X)) \prec \infty$

(c) $\theta_k(x) - g(x) \neq 0 \quad \forall x \in (\alpha, \beta)$ Then

$$E(g(Y_k)|Y_k \succ t) = \theta_k(t) \quad \forall t \in (\alpha, \beta) \tag{12}$$

Iff

1. $\theta_k(\cdot)$ is a differentiable function
2. $G(t) = \exp - \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{(n-k+1)[\theta_k(x) - g(x)]} dx = \left[\frac{\theta_k(\alpha) - g(\alpha)}{\theta_k(t) - g(t)}\right]^{\frac{1}{n-k+1}} \exp - \int_\alpha^t \frac{g^{\backslash}(x)}{(n-k+1)[\theta_k(x) - g(x)]} dx$
3. The function $Q_k(t) = \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{(n-k+1)[\theta_k(x) - g(x)]} dx$ is a non-negative increasing function.
4. $\lim_{t \to \beta^-} \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{(n-k+1)[\theta_k(x) - g(x)]} dx = \infty$
5. $\lim_{t \to \beta^-} \int_\alpha^t \frac{g^{\backslash}(x)}{\theta_k(x) - g(x)} dx = \infty$
6. $\lim_{t \to \beta^-} \theta_k(t) \exp - \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{\theta_k(x) - g(x)} dx = 0$

Proof Necessity

The density function of $Y_k|Y_k \succ t$ [6] is given by :

$$f_n(x) = \frac{n-k+1}{[G(t)]^{n-k+1}} f(x)[G(x)]^{n-k} \ , \quad x \in (t, \beta) \tag{13}$$

Therefore [Eq-12] is equivalent to the following equation:

$$(n-k+1)\int_t^\beta g(x) f(x)[G(x)]^{n-k} \quad dx = \theta_k(t)[G(t)]^{n-k+1} \tag{14}$$

To prove (1), we note that the continuity of the functions $f(\cdot), g(\cdot)$ and G(·) implies the continuity of their product. Therefore, the function $g(x)f(x)[G(x)]^{n-k}$ is integrable. Hence the left hand side of [Eq-14] is differentiable. This means that the right hand side of [Eq-14] is also differentiable. But the function G(·) is differentiable .Therefore the function $\theta_k(\cdot)$ must be differentiable.

To Prove (2), differentiate [Eq-14] with respect to t, recalling that $f(t) = -G^{\backslash}(t)$ and dividing both sides by $[G(t)]^{n-k}$ one gets:

$$-(n-k+1)g(t)f(t) = -(n-k+1)\theta_k(t)f(t) + \theta_k^{\backslash}(t)G(t)$$

Adding to both sides the term $(n-k+1)\theta_k(t)f(t)$ and dividing the result by $(n-k+1)[\theta_k(t) - g(t)]G(t)$ we get: $\frac{f(t)}{G(t)} = \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]}$

Integrating both sides with respect to t from α to x, noting that $\lim_{x \to \alpha^+} G(x) = 1$ and performing some elementary computation, one gets:

$$G(x) = \exp - \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} dt = \left[\frac{\theta_k(\alpha) - g(\alpha)}{\theta_k(t) - g(t)}\right]^{\frac{1}{n-k+1}} \exp - \int_\alpha^x \frac{g^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} dt$$

Proof of (3)

The virtue that G(x) is a survival function implies that:

(a) $0 \le G(x) \le 1$ (b) $G(x)$ is a decreasing function.

It is easy to see (using condition 2) that:

$$Q_k(x) = -\ln G(x) \ge 0, \quad \forall x \in (\alpha, \beta).$$

Also, differentiating the last equation with respect to x, one gets:

$$Q_k^{\backslash}(x) = \frac{-G^{\backslash}(x)}{G(x)} \ge 0, \ \forall x \in (\alpha, \beta).$$

Hence, $Q(x)$ is a non-negative increasing function.

Proof of (4)

The virtue that $G(x)$ is a survival function implies that $\lim_{x \to \beta^-} G(x) = 0$ Therefore, on taking the limit of $G(x)$ defined in condition 2 as $x \to \beta^-$ and making use of the continuity of the exponential function, we conclude that:

$$\lim_{x \to \beta^-} \exp - \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} dt = \exp - \lim_{x \to \beta^-} \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} dt = 0$$

Hence $\lim_{x \to \beta^-} \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} dt = \frac{1}{(n-k+1)} \lim_{x \to \beta^-} \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{\theta_k(t) - g(t)} dt = \infty$

Since n-k+1 is a positive integer,

we conclude that $\lim_{x \to \beta^+} \int_\alpha^x \frac{\theta_k^{\backslash}(t)}{\theta_k(t) - g(t)} = \infty$

Proof of (5)

It is easy to see that (using condition 2 and the assumptions imposed on the functions $\theta_k(\cdot)$ and $g(\cdot)$

$$\lim_{t \to \beta^-} [\theta_k(t) - g(t)][G(t)]^{n-k+1} = \lim_{t \to \beta^-} [\theta_k(\alpha) - g(\alpha)] \exp - \int_\alpha^t \frac{g^{\backslash}(x)}{\theta_k(x) - g(x)} dx =$$

$$[\theta_k(\alpha) - g(\alpha)] \lim_{t \to \beta^-} \exp - \int_\alpha^t \frac{g^{\backslash}(x)}{\theta_k(x) - g(x)} dx = 0$$

Since $\theta_k(\alpha) - g(\alpha)$ is a non-zero finite number, it follows that (making use of the continuity of the exponential function):

$$\lim_{t \to \beta^-} \exp - \int_\alpha^t \frac{g'(t)}{\theta_k(x) - g(x)} dx = \exp - \lim_{t \to \beta^-} \int_\alpha^t \frac{g'(x)}{\theta_k(x) - g(x)} = 0$$

This implies that $\lim_{t \to \beta^-} \int_\alpha^t \frac{g'(x)}{\theta_k(x) - g(x)} dx = \infty$

Proof of (6)

[Eq-14] implies that $\lim_{t \to \beta^-} \theta_k(t)[G(t)]^{n-k+1} = 0$

Therefore $\lim_{t \to \beta^-} \theta_k(t) \exp \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{\theta_k(x) - g(x)} dx = 0$

Sufficiency

At first, we note that the conditions 1, 3, 4, and 5 implies that the function G(x) defined by condition 2 is a survival function of some random variable X. thus differentiating $G(t)$ with respect to t, one gets :

$$f(t) = \frac{\theta_k^{\backslash}(t)}{(n-k+1)[\theta_k(t) - g(t)]} \exp - \int_\alpha^t \frac{\theta_k^{\backslash}(x)}{(n-k+1)[\theta_k(x) - g(x)]} dx = \frac{\theta_k^{\backslash}(t) G(t)}{(n-k+1)[\theta_k(t) - g(t)]}$$

Multiplying the last equation by $(n-k+1)[\theta_k(t) - g(t)][G(t)]^{n-k}$ and performing some elementary computation, one gets:

$$-(n-k+1)g(t)f(t)[G(t)]^{n-k} = -(n-k+1)\theta_k(t)f(t)[G(t)]^{n-k} + \theta_k^{\backslash}(t)[G(t)]^{n-k+1} = \frac{d}{dt}\theta_k(t)[G(t)]^{n-k+1}$$

Integrating both sides with respect to t from x to β, recalling that $\lim_{t \to \beta^-} \theta_k(t)[G(t)]^{n-k+1} = 0$ and dividing both sides by $[G(t)]^{n-k+1}$ one gets: $E(g(Y_k)|Y_k \succ x) = \theta_k(t)$

Remarks (D)

It is clear (from [Eq-13]) that $E(g(Y_n)|Y_n \succ t) = E(g(X)|X \succ t).$

Therefore, we can say that Theorem (D) generalizes Theorem 3 of Su and Huang [24].

Example (B): Let X be an absolutely continuous random variable with cdf $F(\cdot)$. Denote by $Y_1 \prec Y_2 \prec \cdots \prec Y_n$ the order statistics associated with a random sample of size n from $F(\cdot)$. We can show that X follows the uniform distribution with parameters α, β iff for any natural number k where $1 \le k \le n$, the following condition is satisfied:

$$E(Y_k|Y_k \succ t) = \frac{(n-k+1)t+\beta}{n-k+2}\ , \quad t \in (\alpha, \beta).$$

It is easy to see that both of the functions $\theta_k(t) = \frac{(n-k+1)t+\beta}{n-k+2}$ and $g(t) = t$ satisfies all conditions stated in Theorem (D).

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