Journal of Statistics and Mathematics
ISSN: 0976-8807 & E-ISSN: 0976-8815, Volume 3, Issue 3, 2012, pp.-130-133.
Available online at http://www.bioinfo.in/contents.php?id=85

SOME CHARACTERISTIC PROPERTIES OF THE EXPONENTIAL FAMILY

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Received: November 05, 2012; Accepted: December 03, 2012

Abstract- In this paper, two recurrence relations characterizing a certain class of distribution family are presented. The first one is a recurrence relation between conditional moments of h(X) given X < y. The second is a relationship between the conditional moments $E(h^m(Y_k)|Y_k > t)$ and $E(h^{m-1}(Y_k)|Y_k > t)$ where Y_k is the k[th] order statistic from a sample of size n. Finally the concept of conditional variance of $h(Y_k)$ given Y_k > t is used to characterize this family. Some results concerning Modified Weibull, Weibull, Rayleigh, exponential, Linear failure rate, 1st type Pearsonian distributions, Burr, Pareto, Power and uniform distributions are obtained as special cases.

Keywords- Characterization, truncated moments, conditional variance, order statistics, recurrence relations, Modified Weibull, Linear failure rate, Pearson distribution of the first type, Burr, and Power distributions.

Citation: Ali A. A-Rahman (2012) Some Characteristic Properties of the Exponential Family. Journal of Statistics and Mathematics, ISSN: 0976-8807 & E-ISSN: 0976-8815, Volume 3, Issue 3, pp.-130-133.

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Introduction

Characterization theorems play a vital role in many fields such as mathematical statistics, reliability and actuarial science. They form an essential tool of statistical inference as they provide characteristic properties of distributions that enable researchers to identify the particular models. Some excellent references are, e.g., Azlarov and Volodin [6], Galambos and Kotz [11], Kagan, Linnik, and Rao [15] and Mchlachlan and Peel [19], among others.

Different methods have been used to identify several types of distributions. Gupta [13], Ouyang [23], Talwalker [26] and Elbatal et. al.[9], among others have used the concept of right truncated moments to identify some different probability distributions such as Weibull, Burr, Pareto, exponential, power and uniform distributions. In fact characterizations by right truncated moments is very important in practical since, e.g., in reliability studies some measuring devices may be unable to record values greater than time t. Actually there are some measuring devices that can't be able to record values smaller than time t. This has motivated several authors to deal with the problem of characterizing distributions using left truncated moments, see, e.g., Osaki and Li [22], Ahmed [2],Dimaki and Xekalaki [7], Navaro et. al. [20] and Gupta [13]. Furthermore, characterizations of some particular distributions based on conditional variance and conditional moments of order statistics have been considered by several authors such as Khan and Beg [16], Pakes et al. [24], Fakhry [10], El-Arishi [8], Obretenov [21], Wu and Ouyang [27], Ahsanullah and Nevzorov [4] and Govindarajulu [12], among others.

Let X be a continuous random variable with distribution function F (x) defined by :

$$F(x) = 1 - \exp - \left[\frac{h(x) - h(a)}{c} \right], x \in (a, b) \qquad (1)$$

Such that:

1. c is a positive constant.
2. h(x) is a real valued differentiable function defined on (a, b) with:

(a) $\lim_{x \to a^+} h(x) = h(a)$

(b) $\lim_{x \to b^-} h(x) = \infty$

(c) E (h(X)) exists and finite.

It is easy to see that several well known distributions (like Modified Weibull, Linear failure rate, Rayleigh, exponential, Burr, Pareto,… etc) arise from the above family by suitable choices for the function h(x), the value of the parameter c and the domain (a, b).

Main Results

Recurrence relations are very interesting in Statistics. They are used to reduce the number of operations required to obtain a general form for the function under study. Moreover, recurrence relations together with some initial conditions define functions uniqeually. This has motivated several authors to use this concept to characterize some probability distributions see, e.g., Al-Hussaini et. al. [5], Ahmad [1], Lin [18], Khan, et al. [17] and Fakhry [10].

The following Theorem identifies the distribution defined by (1) in terms of a recurrence relation between conditional moments of

$$h^m(X),\ m = 1,2,\ldots \text{ given } X < y$$

Theorem (A)

Let X be an absolutely continuous random variable with cumulative distribution function F(·), density function f(·), failure rate λ(·) and reversed failure rate μ(·) such that F(a) =0 and F(b) = 1 and F(·) has continuous first order derivative on (a,b) with $F'(x) > 0$ for all x. Then, the random variable X has the distribution defined by [Eq-1] iff for any natural number m, the following recurrence relation is satisfied:

$$E(h^m(X)\,|\,X<y)=\frac{-\mu(y)}{\lambda(y)}h^m(y)+mcE(h^{m-1}(X)\,|\,X<y)+\frac{h^m(a)}{F(y)} \quad (2)$$

Proof Necessity

By definition

$$E(h^m(X)\,|\,X)<y)=\frac{\int_a^y h^m(x)dF(x)}{F(y)}$$

Integrating by parts and using the fact that F(a) = 0, one gets:

$$E(h^m(X)\,|\,X<y)=h^m(y)-\frac{m}{F(y)}\int_a^y h^{m-1}(x)h'(x)F(x)dx$$

Using [Eq-1] to eliminate F(x), one gets:

$$E(h^m(X)\,|\,X<y)=h^m(y)-\frac{m}{F(y)}\int_a^y h^{m-1}(x)\left[1-\exp-\left(\frac{h(x)-h(a)}{c}\right)\right]h'(x)dx=$$

$$h^m(y)-\frac{m}{F(y)}\int_a^y h^{m-1}(x)d\,h(x)+\frac{m}{F(y)}\int_a^y h^{m-1}(x)\exp-\left[\frac{h(x)-h(a)}{c}\right]h'(x)\,dx$$

It is easy to see that

$$h'(x)=\frac{c\,f(x)}{1-F(x)}=\frac{c\,f(x)}{\exp\frac{-1}{c}[h(x)-h(a)]} \quad (3)$$

Therefore,

$$E(h^m(X)\,|\,X<y)=h^m(y)-\frac{h^m(y)}{F(y)}+\frac{h^m(a)}{F(y)}+\frac{m\,c}{F(y)}\int_a^y h^{m-1}(x)\ f(x)\,dx=$$

$$-\frac{[1-F(y)]}{F(y)}h^m(y)+m\,c\,E(h^{m-1}(X)\,|\,X<y)+\frac{h^m(a)}{F(y)}$$

Recalling that $\lambda(y)=\frac{f(y)}{1-F(y)}$ *and* $\mu(y)=\frac{f(y)}{F(y)}$

one gets:

$$E(h^m(X)\,|\,X<y)=-\frac{\mu(y)}{\lambda(y)}h^m(y)+m\,c\,E(h^{m-1}(X)\,|\,X<y+\frac{h^m(a)}{F(y)}$$

Sufficiency

[Eq-2] can be written as an equation of the unknown function F(y) as follows:

$$\int_a^y h^m(x)f(x)dx=-[1-F(y)]h^m(y)+m\,c\int_a^y h^{m-1}(x)f(x)dx+h^m(a)$$

Differentiating both sides with respect to y, cancelling out the term $h^m(y)f(y)$ from both sides, adding to both sides $-m\,c\,h^{m-1}(y)f(y)$ and dividing both sides by $-m\,c[1-F(y)]h^{m-1}(y)$

we get: $\frac{f(y)}{1-F(y)}=\frac{h'(y)}{c}$

Integrating both sides with respect to y from a to x, noting that F(a) = 0 and solving the result for F(x),

one gets: $F(x)=1-\exp-\left[\frac{h(x)-h(a)}{c}\right]$

This completes the proof.

Remarks (A)

1. If we put $\phi(X)=h(X)-h(a)$ and m = 1 in Theorem (A), we obtain Talwalker's result [26].
2. If we put $m=1, h(X)=X, a=0$ *and* $b=\infty$ in Theorem (A), we obtain the result of Elbatal, et. al. [9] concerning the exponential distribution.

The next Theorem gives a recurrence relation between conditional moments of $h^m(Y_k)$ *and* $h^{m-1}(Y_k)$ *given* $Y_k>t$

Theorem (B)

Let X be an absolutely continuous random variable with cumulative distribution function F(·), survival function G(·) and density function f(·). Let $X_1, X_2, \dots, X_n$ be a random sample from F(·). Denote by $Y_1<Y_2<\dots<Y_n$ the corresponding ordered sample. Then, the random variable X has the distribution defined by [Eq-1] iff for any natural number m, the following recurrence relation is satisfied :

$$E(h^m(Y_k)\,|\,Y_k>t)=h^m(t)+\frac{mc}{n-k+1}E(h^{m-1}(Y_k)\,|\,Y_k>t),k=1,2,\dots,n \quad (4)$$

Proof Necessity

The conditional density function of the k^{th} order statistic $Y_k|\ Y_k>t$ (Ahsanullah [3]) is given by:

$$f_n(Y_k\,|\,Y_k>t)=\frac{n-k+1}{[G(t)]^{n-k+1}}f(y)[G(y)]^{n-k}\ y\in(t,b) \quad (5)$$

Therefore

$$E(h^m(Y_k)\,|\,Y_k>t)=\frac{n-k+1}{[G(t)]^{n-k+1}}\int_t^b h^m(y)f(y)[G(y)]^{n-k}\,dy=$$

$$\frac{-1}{[G(t)]^{n-k+1}}\int_t^b h^m(y)\,d[G(y)]^{n-k+1}$$

Integrating by parts, recalling that G(b)=0, we get:

$$E(h^m(Y_k)\,|\,Y_k>t)=h^m(t)+\frac{m}{[G(t)]^{n-k+1}}\int_t^b h^{m-1}(y)\,h'(y)[G(t)]^{n-k+1}\,dy$$

Using [Eq-3] to eliminate h'(y) from the 2nd term, one gets:

$$E(h^m(Y_k)\,|\,Y_k>t)=h^m(t)+\frac{mc}{[G(t)]^{n-k+1}}\int_t^b h^{m-1}(y)f(y)[G(y)]^{n-k}\,dy=$$

$$h^m(t)+\frac{mc}{n-k+1}E(h^{m-1}(Y_k)\,|\,Y_k>t)$$

Sufficiency

[Eq-4] can be written in the following integral form:

$$(\mathrm{n-k+1})\int_t^b \mathrm{h}^m(y)f(y)[G(y)]^{n-k}dy=\mathrm{h}^m(t)[G(t)]^{n-k+1}+m\,c\int_t^b h^{m-1}(y)f(y)[G(y)]^{n-k}\,dy$$

Differentiating both sides with respect to t, recalling that f(t) = -G'(t), cancelling out $(n-k+1)h^m(t)f(t)[G(t)]^{n-k}$ from both sides, adding $m\,c\,h^{m-1}(t)f(t)[G(t)]^{n-k}$ to both sides, then dividing both sides by $m\,c\,h^{m-1}(t)[G(t)]^{n-k+1}$

one gets: $\frac{f(t)}{G(t)}=\frac{h'(t)}{c}$

Integrating both sides with respect to t from a to x, recalling that G (a) = 1 and solving the result for G(x),

we get: $G(x)\ =\ \exp\ -\left[\frac{h(x)-h(a)}{c}\right], x\in(a,b)$

This completes the proof.

Remarks (B)

1. If we put k=n in Theorem (B), we obtain a recurrence relation concerning the maximum. Moreover, if we set k = n in [Eq-5], we get: $E(h^m(Y_n)\,|\,Y_n>t)=E(h^m(X)\,|\,X>t)$
2. Theorem (B) generalizes the result of Fakhry[10]. To see this, put k = n.
3. Theorem (B) generalizes the result of Ouyang [23].To see this, put m =1 and k = n.
4. Theorem (B) generalizes the result of Hamdan [14]. To see this put m = 1, k = n and $\phi(X)=h(X)-h(a)$
5. Theorem (B) generalizes the result of Shanbhag [25]. To see this set $\mathrm{m}=1, \mathrm{k}=\mathrm{n}, \mathrm{h(X)}=\mathrm{X}, \mathrm{a}=0 \text{ and } \mathrm{b}=\infty$
6. If we put k=1 in Theorem (B), we obtain a recurrence relation concerning the minimum.
7. If we put n=2r+1 and k= r+1 in Theorem (B), we obtain a recurrence relation concerning the median.

We end our results by the following theorem which identifies the distribution defined by (1) in terms of the conditional variance of some function of the k^{th} order statistic.

Theorem (C)

Let X be an absolutely continuous random variable with density function f (·) , survival function G(·) and distribution function F(·) such that F(a) = 0 and F(b) = 1. Let $X_1, X_2, \ldots, X_n$ be a random sample from F(·). Denote by $Y_1 < Y_2 < \ldots < Y_n$, the corresponding ordered sample. Then, the random variable X has the distribution defined by (1) iff

$$V(h(Y_k)\,|\,Y_k>t)=\frac{C^2}{(n-k+1)^2},\ 1\le k\le n \tag{6}$$

Proof Necessity

From Theorem (B), we have:

$$E(h(Y_k)\,|\,Y_k>t)=h(t)+\frac{C}{n-k+1}$$

$$E(h^2(Y_k)\,|\,Y_k>t)=h^2(t)+\frac{2C}{n-k+1}E(h(Y_k)\,|\,Y_k>t)=$$

$$h^2(t)+\frac{2C}{n-k+1}\left[h(t)+\frac{C}{n-k+1}\right]=h^2(t)+\frac{2C}{n-k+1}h(t)+\frac{2C^2}{(n-k+1)^2}$$

Therefore,

$$V(h(Y_k)\,|\,Y_k>t)=E(h^2(Y_k)\,|\,Y_k>t)-[E(h(Y_k)\,|\,Y_k>t)]^2=\frac{C^2}{(n-k+1)^2}$$

Sufficiency

[Eq-6] can be written in integral form as follows:

$$\frac{n-k+1}{[G(t)]^{n-k+1}}\int_t^b h^2(y)f(y)[G(y)]^{n-k}\,dy-\frac{[n-k+1]^2}{[G(t)]^{2(n-k+1)}}$$

$$\left[\int_t^b h(y)f(y)[G(y)]^{n-k}\,dy\right]^2=\frac{C^2}{(n-k+1)^2}$$

Multiplying both sides by $[G(t)]^{2(n-k+1)}$, differentiating both sides with respect to t and dividing the result by (n-k+1) f(t) $[G(t)]^{n-k}$, one gets:

$$-(\mathrm{n}-\mathrm{k}+1)\int_t^b h^2(y)f(y)[G(y)]^{n-k}\,dy-h^2(t)[G(t)]^{n-k+1}+2(n-k+1)h(t)$$

$$\int_t^b h(y)f(y)[G(y)]^{n-k}\,dy=\frac{-2C^2[G(t)]^{n-k+1}}{(n-k+1)^2}$$

Differentiating both sides with respect to t, cancelling out the term $2(n\text{-}k+1)\,h^2(t)f(t)[G(t)]^{n-k}$ from both sides then dividing the result by 2h'(t), one gets:

$$-h(t)\,[G(t)]^{n-k+1}+(n-k+1)\int_t^b h(y)f(y)[G(y)]^{n-k}\,dy=\frac{c^2[G(t)]^{n-k}}{n-k+1}\,\frac{f(t)}{h`(t)}$$

Again, differentiating both sides with respect to t, cancelling out the term $(n\text{-}k+1)\,h(t)\,f(t)[G(t)]^{n-k}$ from both sides, adding to both sides, $h'(t)[G(t)]^{n\text{-}k+1}$ then dividing both sides by $h'(t)[G(t)]^{n-k-1}$

one gets: $\dfrac{C^2}{n-k+1}\dfrac{G(t)}{h'(t)}\dfrac{d}{dt}\left[\dfrac{f(t)}{h'(t)}\right]-\dfrac{C^2(n-k)}{n-k+1}\dfrac{[f(t)]^2}{[h'(t)]^2}+[G(t)]^2=0$

Recalling that f(t) = -G'(t), the last equation takes the form:

$$\frac{C^2}{n-k+1}\frac{G(t)}{h'(t)}\frac{d}{dt}\left[\frac{G'(t)}{h'(t)}\right]+\frac{C^2(n-k)}{n-k+1}\frac{[G'(t)]^2}{[h'(t)]^2}-[G(t)]^2=0$$

The above equation is satisfied for every 1 ≤ k ≤ n. Therefore, on setting k = n,

we get: $$\frac{C^2}{h'(t)}\frac{d}{dt}\left[\frac{G'(t)}{h'(t)}\right]-G(t)=0 \tag{7}$$

Let z = h(t)–h(a), then $G'(t)=\dfrac{d\,G(t)}{dt}=\dfrac{d\,G(z)}{d\,z}\dfrac{d\,z}{dt}=h'(t)\,G'(z)$

Therefore,

$$\frac{d}{dt}\left[\frac{G'(t)}{h'(t)}\right]=\frac{d}{dt}G'(z)=\frac{dG'(z)}{dz}\frac{d\,z}{dt}=h'(t)\,G''(z)$$

Hence, [Eq-7] takes the form: $C^2G''(z)-G(z)=0$

This is a second order differential equation with constant coefficient, its solution is known to be: $G(z)=A\exp-\left(\dfrac{z}{c}\right)+B\,\exp\left(\dfrac{z}{c}\right)$

Therefore,

$$G(t)=A\exp-\left[\frac{h(t)-h(a)}{c}\right]+B\exp\left[\frac{h(t)-h(a)}{c}\right]$$

The facts that $\lim_{t\to b^-} h(t)=\infty$ and G(b) =0 give B = 0, while the facts that $\lim_{t\to a^+} h(t)=h(a)\ and\ G(a)=1\ give\ A=1$

Hence, $$G\,(t)\ =\ \exp\ -\left[\frac{h(t)-h(a)}{c}\right]$$

The proof is complete.

Remarks (C)

1. If we put k = n in Theorem (C), we obtain a characterization in terms of the conditional variance of the maximum. Furthermore, if we set k = n in [Eq-5],

we get: $V(h(Y_n)\,|\,Y_n>t)=V(h(X)\,|\,X>t)$ Therefore, we can say that Theorem (C) generalizes the result of Fakhry [10].

2. If we put k = n, h(x) =x^λ, where λ is a positive parameter, a = 0 and b =∞, we get the result of Khan and Beg [16].
3. If we put k = 1 in Theorem (C), we obtain a characterization in terms of the conditional variance of the minimum.
4. If we put n =2r+1 and k = r+1 in Theorem (C), we obtain a characterization in terms of the conditional variance of the median.
5. $V(h(Y_1)\,|\,Y_1>t)=\dfrac{C^2}{n^2}<V(h(Y_j)\,|\,Y_j>t)$ for every j, 2 ≤ j ≤ n.

General Comments (A)

In all of the foregoing Theorems, some special results can be picked out for some well known distributions by suitable choices for the function $h(X)$, the value of the parameter c and the domain (a,b) as follows:

1. If we set $h(X) = \alpha X + \beta X^{\lambda}$, where $\lambda > 0$ and $\alpha, \beta \geq 0$,such that $\alpha + \beta > 1$, c=1, a=0 and $b = \infty$, we obtain results concerning modified Weibull distribution with non-negative parameters α and β and positive parameter λ. For λ= 2, we have results concerning the linear failure rate distribution with positive parameters α and β(see [28]). For α = 0, we have results concerning the Weibull distribution with parameters β, λ > 0. For β = 0, we have results concerning the exponential distribution with parameter α > 0. For α = 0 and λ = 2, we have results concerning Rayleigh distribution with parameter β > 0.
2. If we put $h(X) = -\ln\left[\frac{r-X}{r-\mu}\right], a=\mu, b=r$ we obtain results concerning the first type Pearson distribution with parameters μ, r and $1/c$
3. If we put $h(X) = \ln X, a=1 \; and \; b=\infty$ we obtain results concerning Pareto distribution of the 1st type with parameter $1/c$
4. If we set $h(X) = -\ln(1-X)$, a = 0, b = 1, we obtain results concerning beta distribution with parameters 1, $1/c$.
5. If we Set $h(X) = \ln(1+x^r)$, r > 0, a =0 and b = ∞, we obtain results concerning the 2nd type Burr distribution with parameters r, $1/c$.
6. If we Set $h(X) = -\ln(1 - x^a)$, α > 0, c = 1, a =0 and b = 1, we obtain results concerning the power distribution with parameter α. For α = 1, we have results concerning the uniform distribution.

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