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BIFURCATION ANALYSIS OF (2+1) DIMENSIONAL KONOPELCHENKO-DUBROVSKY SYSTEM PRESENTED BY A FRACTIONAL DIFFERENTIAL EQUATION

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Abstract- In this article, we discuss the analytical and numerical bifurcation of (2+1) Dimensional Konopelchenko-Dubrovsky equation presented by a fractional differential equation. The bifurcation parameters and the corresponding phase portraits are illustrated. These results show that the qualitative behaviors of phase portraits are very sensitive to the fractional order derivative.
Keywords- Fractional differential equations; (2+1) Dimensional Konopelchenko-Dubrovsky system of equations; bifurcation theory; non-standard numerical schemes.

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Introduction

Konopelchenko and Dubrovsky (KD) introduced their (2+1)-dimensional model equation in 1984 [6]. Extensive efforts have been devoted to solving and analyzing the qualitative behavior of the solutions and to discussing the bifurcation behavior of this equation for varying parameters. While no unified method to attack this nonlinear problem has appeared, many appealing methods have been utilized to determine exact solutions of the KD equation. The Exp-function method [1], the inverse scattering transform method [6], the homogenous balance method [12] and the improved *tanh* function method [4,9,10] are among these. The existence of varying parameters in this non-linear partial differential equation makes it very rich as a dynamical system. Recently, He [5] extensively analyzed the bifurcation behavior of the KD equation, determining bifurcation parameter sets and their corresponding phase portraits. By using the bifurcation method of planar dynamical systems [2-4, 7], this has led to exact explicit parametric representations of solitary wave solutions, kink (anti-kink) wave solutions and periodic wave solutions of the (2+1) KD equation.

Recall that a local operator, such as an integer order differential equation, has the property only its present state determines its next state, so this operator is indifferent to its history. Conversely, a non-local property is one in which next state of one system depends not only upon its current state but also upon all of its historical states starting from the initial time. The latter more closely reflects reality and is a primary reason why Fractional Differential Equations (FDEs) are increasingly applied to dynamical systems. Thus, we were motivated to conduct a bifurcation analysis of the (2+1) KD equation presented by an FDE and investigate the sensitivity of its various bifurcation phenomena to the fractional order derivative of the equation.

In this article, we first convert the KD equation presented by an FDE to a system of ODEs with fractional order. Then we determine the set of bifurcation parameters by using the theory of dynamical systems whereby phase portraits can be detected and illustrated. Among these phase portraits, saddle node, limit cycle, cusp and homoclinic orbits are illustrated for various values of fractional order derivative. In discretizing the transferred ODE with fractional order, we have applied the Mickens non-standard discretization scheme [12] to the Grunwald-Letnikov discretization process. This non-standard scheme, in the contest of a KD non-linear fractional differential equation, leads to faster convergence and more accurate results as compared to standard alternative methods.

KD Equation Presented by FDE

We consider the KD equation presented by an FDE in time derivative

$$\begin{cases} D_t^{\alpha}u - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0, \\ u_y = v_x, \end{cases} \tag{1}$$

where *a*, *b* are real parameters, $D_t^{\alpha}\mathbf{x}(t) = J^{n-\alpha}D_t^n\mathbf{x}(t)$ is the n^{th} order Riemann-Liouville integral operator defined by $J^n\mathbf{x}(t) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1}\mathbf{x}(\tau)d\tau,$ with $0 < \alpha \le 1$ and $D_t^n(.)$ being ordinary derivative of order *n* for time $t > 0.$

Following the procedure in [5] for ODE case, we let $\xi = x + ky - ct$, with *c* as a wave speed and *k* as a parameter. It follows that $u(x,y,t) = \phi(\xi)$ and $v(x,y,t) = \psi(\zeta)$. Substituting this transformation into the system (1) yields,

$$\begin{cases} -cD_{\xi}^{\alpha}\phi - \phi_{\xi\xi\xi\xi} - 6b\phi\phi_{\xi} + \frac{3}{2}a^2\phi^2\phi_{\xi} - 3k\psi \\ +\frac{3}{2}ak\phi^2 = 0 \\ k\phi_{\xi} = \psi_{\xi}. \end{cases} \tag{2}$$

Integrating both sides of system (2) with respect to ξ leads to

$$\begin{cases} -cI^1D_{\xi}^{\alpha}\phi - \phi_{\xi\xi} - 3b\phi^2 + \frac{1}{2}a^2\phi^3 - 3k^2\psi \\ +\frac{3}{2}ak\phi^2 + 3ag_1\phi - g_2 = 0, \\ k\phi + g_1 = \psi. \end{cases} \tag{3}$$

Here, g_1 and g_2 are integral constants. Now we let $d = \frac{3ak - 6b}{a^2}$, $f = -\frac{-2g_2}{a^2}$, $g = -\frac{2c}{a^2}$, $e = \frac{6ag_1 - ck^2}{a^2} - g$, $\phi_{\xi} = Y$ and $I^1D_{\xi}^{\alpha}\phi = Z$. Then (3) reads

$$\begin{cases} Y_{\xi} = \frac{1}{2}a^2\left[\phi^3 + d\phi^2 + e\phi + f - \frac{2c}{a^2}Z\right] \\ Z_{\xi} = D_{\xi}^{\alpha}\phi \\ \phi_{\xi} = Y. \end{cases} \tag{4}$$

Obviously, for $\alpha = 1$ system (4) becomes

$$\begin{cases} Y_{\xi} = \frac{1}{2}a^2\left[\phi^3 + d\phi^2 + e\phi + f\right] \\ \phi_{\xi} = Y, \end{cases} \tag{5}$$

with some modification in parameter e. As stated in [5], the Hamiltonian function of system (5) is $H(\phi,Y) = \frac{1}{2}Y^2 - \frac{1}{2}a^2\left(\frac{1}{4}\phi^4 + \frac{1}{3}d\phi^3 + \frac{1}{2}e\phi^2 + f\phi\right)$ and the phase portrait, $H(\phi,Y) = h$, define by system (5) determines all solutions of the system (1) in the form of ODE. In this case, one can investigate the bifurcation sets and phase portraits of the system (5) in (ϕ,Y)-phase plane as parameter changes. To do this the first step is to find the fixed points of system (5). Obviously, system (5) has three fixed points, say $(\phi_1,0)$, $(\phi_2,0)$ and $(\phi_3,0)$. Note that, the fixed points of system (4) in (ϕ,Y)-phase plane can be similarly determined with zero Y coordinates and some constants change in ϕ_i corresponding to the changes in e and f. Thus, the fixed points of system (4) can be obtained from those of system (5) by a constants transformation in ϕ direction. Now, at any of these fixed points, the Jacobean of the left hand side equations in (5) has the form $J(\phi_i,0) = -\frac{1}{2}a^2f'(\phi_i)$. So, from theory of dynamical systems if $J(\phi_i,0) < 0$ then the Jacobian has two purely imaginary eigenvalues which means the system has a center around $(\phi_i,0)$. For $J(\phi_i,0) > 0$ we have two eigenvalues with different signs and thus a saddle occurs around $(\phi_i,0)$. Also for $(\phi_i,0) = 0$ and zero Poincare index for $(\phi_i,0)$, there is a cusp. Finally, if the level sets for two different fixed points $(\phi_i,0)$ and $(\phi_j,0)$ have the same values, then the heteroclinic loop occurs. In this article, we will use the same sets of bifurcation parameters as those used in [5] for system (5) in order to investigate the qualitative behavior of these phase portraits for system (4) with varying values of derivative order α. Here, we suppose $a = 1$ and fixed e as a positive number, say 1, and we choose d and f as bifurcation parameters. As discussed in [5], the qualitative behavior of phase portraits for zero or negative values of e is similar to the ones with positive e. So we can vary only d and f. Now, according to the theory of dynamical systems, if we apply the conditions for the saddle, center, cusp and heteroclinic loop to the system (5) the (d,f) plane will be divided in 9 areas by the curves and the lines; $f=\dfrac{-2d^3+9d-(2d^2-6)\sqrt{d^2-3}}{27}$, $f=\dfrac{-2d^3+9d+(2d^2-6)\sqrt{d^2-3}}{27}$, $f=\dfrac{-2d^3+9d}{27}$, $d=-\sqrt{3}$ and $d=\sqrt{3}$ as in Fig. 1. These areas are considered as

$$A0=\left\{(d,f),\ -\sqrt{3}<d<\sqrt{3}\right\},$$

$$A1=\left\{\begin{array}{l}(d,f),\ d>\sqrt{3}, f< \\ \dfrac{-2d^3+9d-(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\},$$

$$A2=\left\{\begin{array}{l}(d,f),\ d>\sqrt{3}, \dfrac{-2d^3+9d-(2d^2-6)\sqrt{d^2-6}}{27} \\ <f<\dfrac{-2d^3+9d}{27}\end{array}\right\},$$

$$A3=\left\{\begin{array}{l}(d,f),\ d>\sqrt{3}, \dfrac{-2d^3+9d}{27} \\ <f<\dfrac{-2d^3+9d+(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\},$$

$$A4=\left\{\begin{array}{l}(d,f),\ d>\sqrt{3}, f> \\ \dfrac{-2d^3+9d+(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\},$$

$$A5=\left\{\begin{array}{l}(d,f),\ d<-\sqrt{3}, f> \\ \dfrac{-2d^3+9d+(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\},$$

$$A6=\left\{\begin{array}{l}(d,f),\ d<-\sqrt{3}, \dfrac{-2d^3+9d}{27}< \\ f<\dfrac{-2d^3+9d+(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\},$$

$$A7=\left\{\begin{array}{l}(d,f),\ d<-\sqrt{3}, \\ \dfrac{-2d^3+9d-(2d^2-6)\sqrt{d^2-6}}{27}< \\ f<\dfrac{-2d^3+9d}{27}\end{array}\right\},$$

$$A8=\left\{\begin{array}{l}(d,f),\ d<-\sqrt{3}, f< \\ \dfrac{-2d^3+9d-(2d^2-6)\sqrt{d^2-6}}{27}\end{array}\right\}.$$

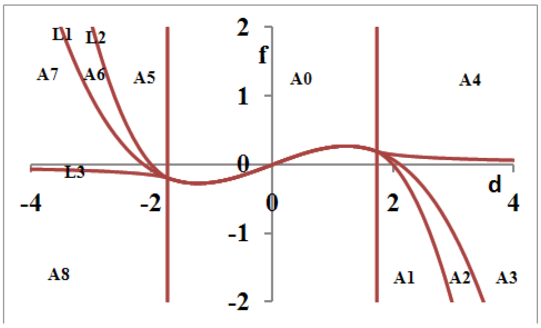


Fig.1- Different regions for the bifurcation set of parameters *d* and *f*.

Discretization and Numerical Results

Here, we shall use the Grunwald-Letnikov method [9,10] to discretize system (4). In this method, fractional derivative is discretized as $D^\alpha \mathbf{x}(t)=\sum_{k=0}^{[t_n/h]} c_k^\alpha \mathbf{x}(t_{n-k})$, where h is the step size, $\left[{t_n}/{h}\right]$ denotes the integer part of ${t_n}/{h}$, $t_n=nh$ and c_k^α are the Grunwald-Letnikov coefficients defined by $c_k^\alpha=h^{-\alpha}(-1)^k\binom{\alpha}{k},\ k=0,1,2,\ldots$. These coefficients can also be evaluated recursively by $c_0^\alpha=h^{-\alpha}$ and $c_k^\alpha=\left(1-\dfrac{1+\alpha}{k}\right)c_{k-1}^\alpha,\ k=1,2,3,\ldots$. Using this method, system (4) discretized as follow.

$$\begin{cases} Z(t_n)=\sum_{k=1}^{N}\left(1-\dfrac{1+\alpha}{k}\right)\phi(t_{n-k}) \\ \phi(t_n)=\phi(t_{n-1})+h\,y(t_{n-1}) \\ y(t_n)=y(t_{n-1})+0.5h \\ \left(\phi(t_{n-1})\phi^2(t_n)+d\phi(t_{n-1})\phi(t_n)+\phi(t_n)+f-2cZ(t_n)\right) \end{cases} \tag{6}$$

Note that in this discretization we have used the non-standard Mickens' method [8] to obtain a stronger result. This means that in the discretization process we have replaced $\phi^3(t_n)$ with $\phi^2(t_{n-1})\phi(t_n)$ or $\phi^2(t_n)$ with $\phi(t_{n-1})\phi(t_n)$. To be consistence with [5] we have chosen the bifurcation parameters *d* and *f* from the following 6 different areas $A=A0\cup A1\cup A4\cup A5\cup A8$, $B=L1\cup\left\{(d,f)/|d|>\sqrt{3}\right\}$, $C=A2\cup A7$ $D=L2\cup\left\{(d,f)/|d|>\sqrt{3}\right\}$, $E=A3\cup A6$ and $F=L3\cup\left\{(d,f)/|d|>\sqrt{3}\right\}$. Now, we are ready to solve the discretized system (6) by choosing the bifurcation parameters from these 6 regions with different initial values. We have illustrated the results in Fig. 2. Each figure in the set of Fig. 2 represents the phase portrait for each above 6 different regions of parameters. In these figures, we have taken the order of derivative $\alpha = 0.9$. As we can see in these figures, the qualitative behavior of the saddle points did not show significant change. This is the same for the cusp points. On the other hand, the center points are very sensitive to the value of derivative order α. We have tried to take the

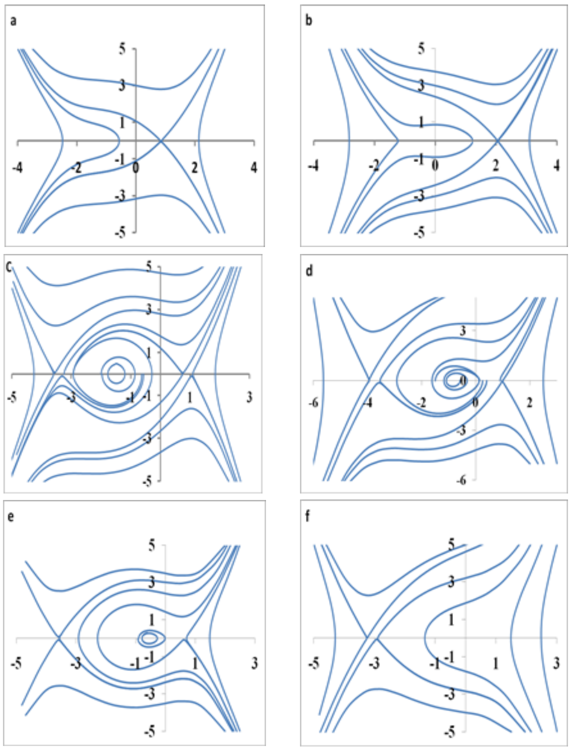
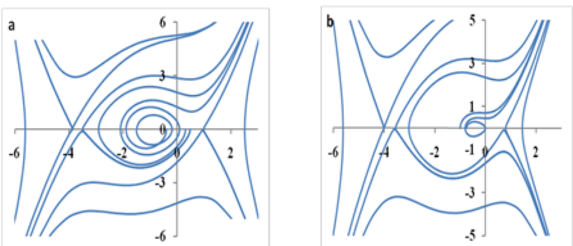


Fig. 2. Phase portriates of KD equation presented by FDE for derivative order $\alpha = 0.9$ and varying values of parameters *d* and *f* in 6 different regions shown in the text.

smallest possible values for the order of derivative α such that the qualitative behavior of phase portraits remain unchanged comparing to the original phase portraits in the case of $\alpha = 1$. We have observed that the different fixed points of the system that exist for the bifurcation values *d* and *f*, will brake done and will be extremely sensitive for the values of $\alpha < 0.8$. For example, as we can see in Fig. 3 even the saddle points related to figures 1-e to 1-f are losing their qualitative behavior for the order derivatives less than 0.8. Of course, these infrastructures are clearer for the center points in these figures (see Fig. 3).



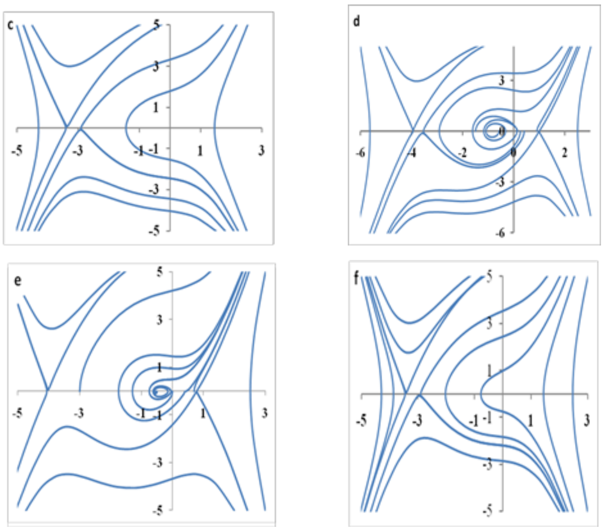


Fig. 3. Phase Portriates of KD equation presented by FDE. Figures a-c are similar to the figures d-f in Fig. 1 for $\alpha = 0.8$. Figures d-f are also similar to figures d-f in Fig. 1 for $\alpha = 0.7$.

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