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THERMAL STRESSES IN A THICK CIRCULAR PLATE WITH INTERNAL HEAT SOURCES

GAHANE T.T., KHALSA L.H. AND KHOBRAGADE N.W.

Department of Mathematics, RTM Nagpur University, Nagpur- 440033, MS, India.
*Corresponding Author: Email- khobragadenw@gmail.com

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Abstract- In this paper, an attempt has been made to study thermoelastic response of a circular plate in general, in which source are generated according to the linear function of the temperature, with boundary conditions of the radiation type, by applying integral transform techniques. The results are obtained in terms of Bessel's functions in the form of infinite series. Numerical calculations are carried out for a particular case of a plate made of Aluminum metal and the results are depicted graphically.
Keywords- Thermoelastic response, circular plate, temperature distribution, thermal stress, integral transform.

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Introduction

During the second half of the Twentieth century, non isothermal problem of the theory of elasticity became increasingly important this is due to their wide application in diverse fields. The high relocation of modern aircraft gives rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure.

Noda [1] studied transient thermal stress problem in a finite circular transversely isotropic solid cylinder subjected to an asymmetrical temperature distribution on a cylindrical surface. The stress fields are found by use of potential functions method. Hasheminejad and Rafsanjani [2] discussed an exact three-dimensional analysis for steady-state dynamic response of an arbitrarily thick, isotropic, and functionally graded plate strip due to the action of a transverse distributed moving line load which is propagating parallel to the infinite simply supported edges of the plate at constant speed based on the linear elasticity theory. The inhomogeneous plate is approximated by a laminate model, for which the solution is expected to gradually approach the exact one as the number of layers increases. The problem solution is derived by using Fourier transformation with respect to a moving reference frame in conjunction with the classical transfer matrix approach entailing the continuity of displacement and stress components at the interfaces of neighboring layers.

Sharma et al.[3] analyzed the propagation of Lamb waves in a homogeneous, transversely isotropic, piezothermoelastic plate, which is stress free, electrically shorted, and thermally insulated (or isothermal). Secular equations for the plate in closed form and isolated mathematical conditions for symmetric and anti symmetric wave mode propagation are derived in completely separate terms. El-Maghraby [4] studied a two-dimensional problem for a half-space. The problem is in the context of the theory of generalized thermoelasticity with one relaxation time. The surface of the half-space is taken to be traction free and the temperature on it is specified. Heat sources permeate the medium. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach.

El-Maghraby [5] discussed the two-dimensional problem for a thick plate whose upper surface is subjected to a known temperature distribution, while the lower surface is laid on a rigid foundation and thermally insulated. Laplace and exponential Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. Sherief and Anwar [6] studied problem of heat conduction of a Thermoelastic cylindrical medium composed of two different materials. The problem has been solved in the framework of the generalized thermoelasticity theory with one relaxation time. The solution is obtained in the Laplace transform domain by using the potential function approach. Numerical inversion formula is used to obtain the corresponding solutions in the physical domain. Noda, Hetnarski and Tanigawa [7] completely discussed all aspect of fundamental thermoelasticity and provides a sound grounding in the fundamental theory of thermal stresses as well as includes a multitude of applications.

Roy Choudhary [8] discussed the normal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face, while the lower face of the plate is kept at zero temperature and the circular edge is thermally insulated. The investigation in the present paper is based on the research papers of Wankhede [9] and Roy Choudhary [8].

Here we have generalized the results of Wankhede [9] and Roy Choudhary [8] and solved the problem of thermoelasticity for a circular plate with the stated boundary conditions. Numerical results are also included. The result presented here may be useful in engineering problem, particularly in determination of the state of strain in thick circular plate constructing foundation of containers for hot gases or liquid in the foundation for furnaces etc.

Statement of the Problem

Consider a thick circular plate of radius *a* and thickness 2*b* defined by $0 \le r \le a, \ -b \le z \le b$. Let the disk be subjected to a transient axisymmetric temperature field on the radius and axial direction of the cylindrical coordinate system. Initially the temperature of the plate is maintained at F(r,z). The third kind condition $Q_0\, g(r,t)/\lambda$ is prescribed over the lower surface (*z* = *b*) and on the upper surface (*z* =- *b*), it is maintained at f(r,t). Under these more realistic prescribed conditions, the transient thermal stresses are required to be determined.

The differential equation governing the displacement potential function $\varphi(r,z,t)$ is given in Noda, et al. as

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = K\tau \tag{2.1}$$

where K $\rightarrow$ restraint coefficient and temperature change $\tau = T - T_i, \ T_i$ is initial temperature, displacement function φ is the Goodier's thermoelastic potential.

The temperature of the plate at time *t* satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{\Omega(r,z,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.2}$$

With the boundary conditions

$$\left[T(r,z,t) + k_1 \frac{\partial T(r,z,t)}{\partial z}\right]_{z=b} = \frac{-Q_0}{\lambda} g(r,t), \quad 0 \le r \le a, \ \textit{for all time } t \tag{2.3}$$

$$\left[T(r,z,t) + k_2 \frac{\partial T(r,z,t)}{\partial z}\right]_{z=-b} = f(r,t), \qquad 0 \le r \le a, \ \textit{for all time } t \tag{2.4}$$

$$\left[T(r,z,t) + \frac{\partial T(r,z,t)}{\partial r}\right]_{r=a} = 0 \ , \tag{2.5}$$

$$T(r,z,t) = F(r,z), \quad at \ t = 0 \tag{2.6}$$

Where *k* is thermal diffusivity of the material of the plate. The displacement function in the cylindrical coordinate system are represented by Michell's function M defined in Noda, et al. [7] as,

$$u_r = \frac{\partial \varphi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \tag{2.7}$$

and

$$u_z = \frac{\partial \varphi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{2.8}$$

The Michell's function must satisfy

$$\nabla^2 \nabla^2 M = 0 \tag{2.9}$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2.10}$$

The component of the stresses are represented by the thermoelastic displacement potential φ and Michel's function *M* as

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \varphi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{2.11}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \varphi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{2.12}$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \varphi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.13}$$

and

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \varphi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.14}$$

For traction free surface, stress functions

$\sigma_{rz} = \sigma_{zz} = 0$ at $z = \pm b$ for thick circular plate. (2.15)

The equations (2.1) – (2.15) constituter the mathematical formulation of the problem under consideration.

Solution of the Problem

Following the general procedure of integral transform we apply finite Marchi-Fasulo transform and Laplace transform to the equation (2.1) to (2.14). Then using their inversion one obtains the expression for temperature distribution as

$$T(r,z,t) = \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^t A_{mn} P_n(z) J_0(r\lambda m)\overline{F}\, e^{-k(\lambda_m^2 + a_n^2)(t-t^1)} dt^1 + \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{\mu_n}\phi \tag{3.1}$$

Where $\phi = L^{-1}(\psi)$,

$\overline{F} = -\{\psi(a) + \psi^1(a)\}$,

$$A_{mn} = \frac{4K}{a}\left(\frac{\lambda_m}{3J_1(a\lambda m) + J_0(a\lambda m)}\right)\frac{1}{\mu_n} \qquad P^2 = a_n^2 + \frac{s}{k},$$

$$A = \frac{-\left[\psi(a) + \psi^1(a)\right]}{J_0(P,a) + J_0^1(P,a)}$$

where μ_m are the positive roots of the transcendental equation $J_0(\mu_m a) = 0.$

Now let us assume Michell's function M, which satisfies condition (2.9) as,

$$M = \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{P_n(z)}{\mu_n}\phi \tag{3.2}$$

Using equations (3.1) and (3.2) in equation (2.1), we obtain the displacement potential function φ as

$$\varphi = \frac{Q_0}{\lambda}\sum_m\sum_n B_{mn}P_n(z)J_0(r\lambda m)B(t) + \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{K\,P_n'(z)}{\mu_n}\phi \tag{3.3}$$

Further using equations (3.1) to (3.3) in equation (2.7), (2.8) and (2.11) to (2.14), we obtain expression for displacement functions and stresses as

$$u_r = \frac{Q_0}{\lambda}\sum_m\sum_n B_{mn}\lambda_m P_n(z)J_1(r\lambda_m)B(t) + \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}(K-1)\frac{P_n'(z)}{\mu_n}\phi' \tag{3.4}$$

$$u_z = \frac{Q_0}{\lambda}\sum_m\sum_n B_{mn}P_n'(z)J_0(r\lambda_m)B(t) + \frac{Q_0}{\lambda}\sum_{n=1}^{\infty}\frac{K\,P_n'(z)}{\mu_n}\phi'$$
$$+2(1-v)\frac{Q_0}{\lambda}\left\{\sum_n\frac{P_n(z)}{\mu_n}\phi'' + \frac{1}{r}\sum\frac{P_n(z)}{\mu_n}\phi' + \sum\frac{P_n''(z)}{\mu_n}\phi\right\} \tag{3.5}$$

$$\sigma_{rr} = \frac{2GQ_0}{\lambda}\left\{\sum_m\sum_n \lambda_m^2 B_{mn}P_n(z)J_1'(r\lambda_m)B(t) + \sum\frac{KP_n'(z)}{\mu_n}\phi''\right.$$
$$-K\left\{\sum_m\sum_n A_{mn}P_n(z)J_0(r\lambda_m)\int_0^t\bar{\phi}\,e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' + \sum\frac{P_n''(z)}{\mu_n}\phi\right\}$$
$$\left.+\frac{v}{r}\sum_n\frac{P_n'(z)}{\mu_n}\phi' + v\sum\frac{P_n'''(z)}{\mu_n}\phi\right\} \tag{3.6}$$

$$\sigma_{\theta\theta} = \frac{2GQ_0}{\lambda}\left\{\frac{1}{r}\sum_m\sum_n \lambda_m B_{mn}P_n(z)J_1(r\lambda_m)B(t) + \frac{1}{r}\sum_n\frac{KP_n'(z)}{\mu_n}\phi'\right.$$
$$-K\left\{\sum_m\sum_n A_{mn}P_n(z)J_0(r\lambda_m)\int_0^t\bar{\phi}\,e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' + \sum\frac{P_n'(z)}{\mu_n}\phi\right\}$$
$$\left.+v\sum_n\frac{P_n'(z)}{\mu_n}\phi'' + v\sum_n\frac{P_n''(z)}{\mu_n}\phi\right\} \tag{3.7}$$

$$\sigma_{zz} = \frac{2GQ_0}{\lambda}\left\{\sum_m\sum_n B_{mn}P_n''(z)J_0(r\lambda_m)B(t) + \sum_n\frac{KP_n'''(z)}{\mu_n}\phi'\right.$$
$$-K\left\{\sum_m\sum_n A_{mn}P_n(z)J_0(r\lambda_m)\int_0^t\bar{\phi}\,e^{-k(\lambda_m^2+a_n^2)(t-t')}dt' + \sum\frac{P_n''(z)}{\mu_n}\phi\right\}$$
$$\left.+(2-v)\sum\frac{P_n'(z)}{\mu_n}\phi'' + \frac{1}{r}\sum\frac{P_n'(z)}{\mu_n}[L^{-1}(\psi)]' + (1-v)\sum_n\frac{P_n'''(z)}{\mu_n}\phi\right\} \tag{3.8}$$

$$\sigma_{rz} = \frac{2GQ_0}{\lambda}\left\{\sum_m\sum_n \lambda_m B_{mn}P_n(z)J_1(r\lambda_m)B(t) + \sum_n\frac{KP_n'(z)}{\mu_n}\phi'\right.$$
$$\left.+(1-v)\left\{\frac{1}{r}\sum_n\frac{P_n(z)}{\mu_n}\phi'' + \sum_n\frac{P_n'''(z)}{\mu_n}\phi'''\right\} - v\sum\frac{P_n'''(z)}{\mu_n}\phi'\right\} \tag{3.9}$$

where

$$A_{mn} = \frac{4K}{\mu_n a}\left[\frac{\lambda_m}{3J_1(a\lambda_m) + J_0(a\lambda_m)}\right], \quad B_{mn} = K\,A_{mn}, \quad B(t) = \int\left(\int_0^t\bar{\bar{\phi}}\,e^{-k(\lambda_m^2+a_n^2)(t-t')}dt'\right)dt$$

Special Case and Numerical Results

Setting

$$F(r,z) = \delta(r - r_0) \times (z-b)^2 \times (z+b)^2 e^{-t} \tag{4.1}$$

where r is the radius of the plate and δ is the Dirac – delta function.

$$\Rightarrow \bar{F} = 8(k_1 + k_2)J_0(\mu_m r_0)\left[\frac{a_n b\cos^2(a_n b) - \cos(a_n b)\sin(a_n b)}{a_n{}^2}\right]e^{-t} \tag{4.2}$$

The plate is thick due to the one –fifth thickness of the largest dimension. Hence a = 5m, b =1m, time is in seconds and for convenience, we set,

$$\alpha = \frac{Q_0}{\lambda}, \quad \beta = \frac{2GQ_0}{\lambda}$$

W h e r e $\xi_1 = 0.4809$, $\xi_2 = 1.1040$, $\xi_3 = 1.7307$, $\xi_4 = 2.3583$, $\xi_5 = 2.9861$ and $\xi_6 = 3.6142$ are the positive roots of the transcendental equation $J_0(5\xi) = 0$ for a = 5 as in [10]. These values are used top evaluate temperature, displacement function and stresses given by equations (3.1) to (3.9) which are illustrated numerically and shown in the figure. Figures 1-4 are the graphical representations of the temperature, displacement function and stresses versus ***z*** at different times.

Material Properties

The numerical calculation has been carried out for a thick circular plate with the material properties as,

Materials	K Btu/ hr ft^0 F	*Cp* Btu/ lb ^{0}F	*p* lb/ft^3	α ft^2/ hr	λ 1/F	*E* GP$_a$	*v*
Aluminum(Al)	117	0.208	169	3.33	12.84 X 10^{-6}	70	0.35
Copper(Cu)	224	0.091	558	4.42	9.3 X 10^{-6}	117	0.36
Iron(Fe)	36	0.104	491	0.7	6.7 X 10^{-6}	193	0.21
Silver(Ag)	242	0.056	655	6.6	10.7 X 10^{-6}	83	0.37

Dimensions

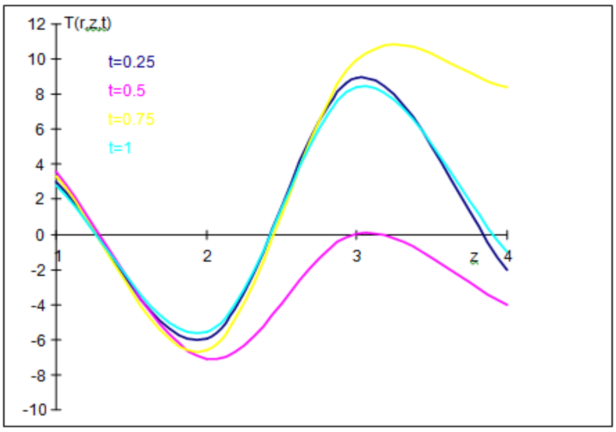
The constants associated with the numerical calculation are taken as radius of a circular plate a =1 ft
thickness of the circular plate b = 0.2 ft
time t = 1 hrs.

Discussion

In this paper we discussed a transient heat conduction problem of a isotropic circular disk under unsteady state temperature field due to internal heat generation within it. As an illustration, we carried out numerical calculations for a thick circular disk made up of aluminum metal and examine the thermoelastic behavior in the state for temperature distribution, displacement and thermal stresses in radial and axial direction.

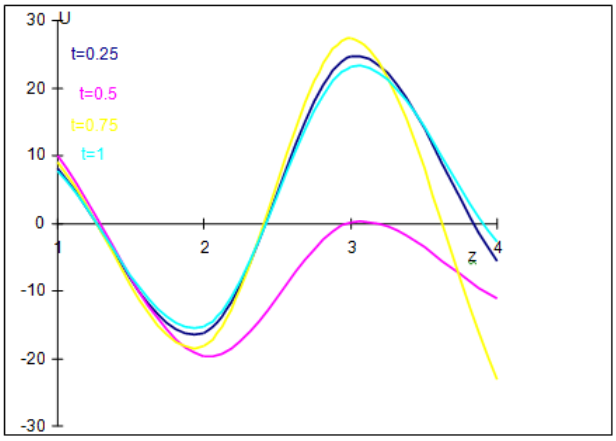
Graphical Analysis

Graph 1 shows that the variation of temperature T (r, z, t) Vs z, it is clear that temperature decreases initially at time t=1, t=0.75, t=0.25 and slightly increasing at z=2.5, the curve behaves like a sinusoidal type. But due to the axisymmetric internal heating at t=0.5 temperature decreases upto zero at z=3.



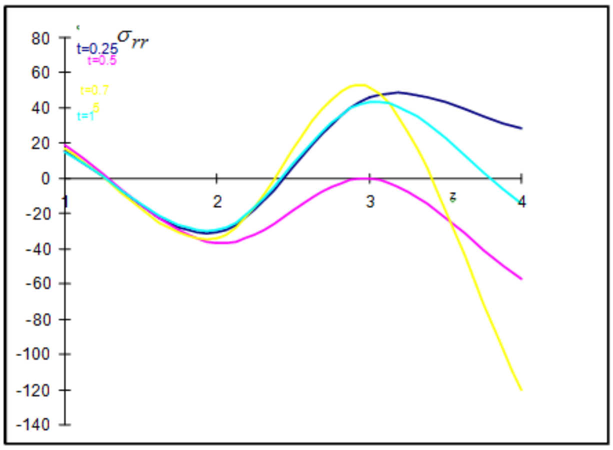
Graph 1- Temperature distribution vs. z for different values of time

Graph 2 shows that the variation of displacement U(r, z, t) Vs z, it is clear that radial displacement decreases initially at time t=1, t=0.75, t=0.25 and slightly increasing at z=2.5 and attain peak value for z=3, again the curve behaves like a sinusoidal type. But due to the axisymmetric internal heating at t=0.5 displacement decreases upto zero at z=3.



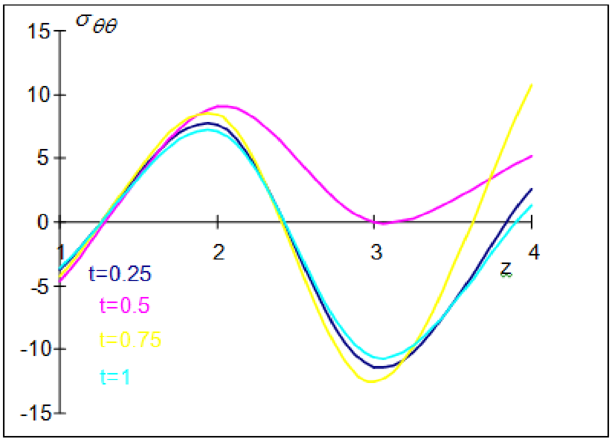
Graph 2- Displacement vs. z for different values of time

Graph3 shows that the variation of radial stresses σ_{rr} Vs z at different values of time, it is clear that radial stresses initially decreases at time t=1, t=0.75, t=0.25 and the start increasing at z=2.5 and attain peak value for z=3, again the curve behaves like a sinusoidal type. But due to the axisymmetric internal heating at t=0.5 temperature decreases upto zero at z=3.



Graph 3- Radial stresses vs. z for different values of time

Graph4 shows that the variation of axial stresses $\sigma_{\theta\theta}$ Vs z at different values of time, it is clear that radial stresses initially increasing at time t=1, t=0.75, t=0.5, t=0.25 and the start decreasing at z=2, but due to the axisymmetric internal heating at t=0.5 stresses increases upto z=2.4 and zero at z=3.



Graph 4- Tangential stresses vs. z for different values of time

Conclusion

In this article, we modify the problem studied by Roy Choudhary [8] and Wankhede [9], and study the thermoelastic problem of a thick circular plate due to partially distributed heat supply and axisymmetric heat supply on the lower plane surface. We develop the analysis for the temperature field by introducing the method of the finite Marchi-Fasulo transform and Laplace transform and determine temperature, displacement function and stresses.

This type of solution is mainly applicable in engineering problems, particularly for a machine subjected to a transient axisymmetric temperature field. Any particular case of special interest can be derived by assigning suitable values to the parameters and functions for the expressions.

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