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SECOND ORDER SYMMETRIC DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING UNDER INVEXITY

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Abstract- In the present paper, a pair of Wolfe type nondifferentiable multiobjective second-order symmetric dual programs involving two kernel functions is formulated. We established weak, strong and converse duality theorems for this pair under invexity assumptions.
Keywords- Nondifferentiable multiobjective programming; Second-order Symmetric duality; Efficiency; Invexity.

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Introduction

Symmetric duality in nonlinear programming was first introduced by Dorn [1]. He introduced the concept of symmetric duality for quadratic problems. Later on Dantzig et al.[2] formulated a pair of symmetric dual nonlinear programs involving convex/concave functions. Mangasarian [3] introduced the concept of second-order duality for nonlinear problems. Mond and Schechter [4] constructed two new symmetric dual pairs in which the objectives contain a support function and are therefore nondifferentiable. Second order symmetric duality for Mond-Weir type duals involving nondifferentiable function has been discussed by Hou and Yang [5].

The symmetric dual problems in the above papers involve only one kernel function. In this paper, we present nondifferentiable symmetric dual multiobjective problems involving two kernel functions.

Prerequisites

We consider the following multiobjective programming problem:

(P) Minimize

$F(x)=\{ F_1(x), F_2(x),..,F_k(x) \}$

Subject to :, $x \in X=\{ x \in R^n \mid G_j(x) \leqq 0, j=1, 2,.., m \}$

Where $G: R^n \to R^m$ and $F: R^n \to R^k$.

For a function $f: R^n \times R^m \to R^k$, let $\nabla_x f\ (\nabla_y f)$ denote the $n \times k\ (m \times k)$ matrix of first order derivatives and $\nabla_{xy} f_i$ denote the $n \times m$ matrix of second order derivatives.

For $a, b \in R^n$,

$a \geqq b \Leftrightarrow a_i \geqq b_i, i = 1,2,...,n,$

$a \geq b \Leftrightarrow a \geqq b$ and $a \neq b,$

$a > b \Leftrightarrow a_i > b_i, i = 1,2,...,n.$

Definition 1

A point $\bar{x} \in X$ is said to be an efficient solution of (P) if there exists no $x \in X$ such that

$F(x) \leq F(\bar{x})$.

Definition 2 [6]

The function F is η -invex at $u \in R^n$ if there exists a vector valued function $\eta : R^n \times R^n \to R^n$ such that

$F(x) - F(u) - \eta^T(x,u)\nabla K(u) \geqq 0, \forall x \in R^n.$

Definition 3 [4]

Let S be a compact convex set in R^n. The support function $s(x|S)$ of S is defined by

$$s(x|S) = \max\{ x^T y : y \in S\}.$$

The subdifferential of $s(x|S)$ is given by

$$\partial s(x|S) = \{ z \in S : z^T x = s(x|S)\}$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$N_S(x) = \{y \in R^n : y^T(z-x) \leqq 0 \text{ for all } z \in S\}.$

It is readily verified that for a compact convex set S, y is in $N_S(x)$ if and only if $s(y|S) = x^T y.$

Wolfe Type Symmetric Duality

We now consider the following pair of Wolfe type second- order nondifferentiable multiobjective programming problems.

Primal (WP)

Minimize

$$f(x,y) + s(x|D)e - (y^T \nabla_y (\lambda^T f(x,y)))e - (y^T(\nabla_{yy}(h^T g(x,y))p))e$$

Subject to

$$y^T \nabla_y (\lambda^T f(x,y)) - w + \nabla_{yy}(h^T g(x,y))p \leqq 0, \tag{1}$$

$$\lambda^T e = 1, \tag{2}$$

$$\lambda > 0, x \geqq 0, w \in R^m. \tag{3}$$

Dual (WD)

Maximize

$$f(u,v) + s(u|E)e - (u^T \nabla_x (\lambda^T f(u,v)))e - (u^T(\nabla_{xx}(h^T g(u,v))q))e$$

Subject to

$$\nabla_x (\lambda^T f(u,v)) + z + \nabla_{xx}(h^T g(u,v))q \geqq 0, \tag{4}$$

$$\lambda^T e = 1, \tag{5}$$

$$\lambda > 0, v \geqq 0, z \in R^n, \tag{6}$$

where

(i) $f : R^n \times R^m \to R^k$ is a twice differentiable function of x and y,

(ii) $g : R^n \times R^m \to R^r$ is a thrice differentiable function of x and y,

(iii) $p \in R^m, q \in R^n, e = (1,...,1)^T \in R^k,$ and

(iv) D and E are compact convex sets in R^n and R^m, respectively.

Any problem, say (WD), in which λ is fixed to be $\bar{\lambda}$ will be denoted by $(WD)_{\bar{\lambda}}$.

Theorem 1 (Weak duality)

Let (x,y,λ,h,w,p) be feasible for (WP) and (u,v,λ,h,z,q) be feasible for (WD). Let

(i) $f(.,v) + ((.)^T z)e$ be $\eta_1 -$ invex at u for fixed v and z,

(ii) $f(x,.) - ((.)^T w)e$ be $\eta_2 -$ invex at y for fixed x and w,

(iii) $\eta_1(x,u) + u \geqq 0, \eta_2(v,y) + y \geqq 0$ and

(iv) $\eta_1^T(x,u)(\nabla_{xx}(h^T g(u,v)q \leqq 0,$
$\eta_2^T(v,y)\nabla_{yy}(h^T g(x,y)p \geqq 0.$

Then

$$f(x,y) + s(x|D)e - (y^T \nabla_y (\lambda^T f(x,y)))e - (y^T(\nabla_{yy}(h^T g(x,y))p))e$$
$$\geqq f(u,v) + s(u|E)e - (u^T \nabla_x (\lambda^T f(u,v)))e - (u^T(\nabla_{xx}(h^T g(u,v))q))e.$$

Proof

Suppose, to the contrary, that

$$f(u,v) + s(u|E)e - (u^T \nabla_x (\lambda^T f(u,v)))e - (u^T(\nabla_{xx}(h^T g(u,v))q))e$$
$$\geq f(x,y) + s(x|D)e - (y^T \nabla_y (\lambda^T f(x,y)))e - (y^T(\nabla_{yy}(h^T g(x,y))p))e.$$

Since $\lambda > 0$ and $\lambda^T e = 1,$ the above vector inequality implies

$$f(u,v) + s(u|E)e - (u^T \nabla_x (\lambda^T f(u,v)))e - (u^T(\nabla_{xx}(h^T g(u,v))q))e$$
$$> f(x,y) + s(x|D)e - (y^T \nabla_y (\lambda^T f(x,y)))e - (y^T(\nabla_{yy}(h^T g(x,y))p))e. \tag{7}$$

From $\eta_1 -$ invexity of $f(.,v) + ((.)^T z)e$, we have

$$f(x,v) + (x^T z)e - f(u,v) - (u^T z)e$$
$$\geqq (\nabla_x^T f(u,v) + z^T e)\eta_1(x,u)^T.$$

Using $\lambda > 0$, we obtain

$$(\lambda^T f)(x,v) + (x^T z) - (\lambda^T f)(u,v) - (u^T z) \geqq \eta_1(x,u)(\nabla_x(\lambda^T f)(u,v) + z). \quad (8)$$

From the dual constraint (4) and hypothesis (iii), it follows that

$$\eta_1(x,u)(\nabla_x(\lambda^T f(u,v)) + z + \nabla_{xx}(h^T g(u,v))q) \geqq -u^T(\nabla_x(\lambda^T f(u,v)) + z + \nabla_{xx}(h^T g(u,v))q). \quad (9)$$

Now inequalities (8), (9) along with hypothesis (iv), yield

$$(\lambda^T f)(x,y) + (x^T z) - (\lambda^T f)(u,v) \geqq -u^T(\nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(h^T g(u,v))q). \quad (10)$$

Similarly by η_2 -invexity of $f(x,.) - ((.)^T w)e$, the primal constraints (1) and hypotheses (iii) and (iv), we obtain

$$(\lambda^T f)(x,y) - (\lambda^T f)(x,v) + (v^T w) \geqq y^T(\nabla_y(\lambda^T f(x,y))) + \nabla_{yy}(h^T g(x,y))p). \quad (11)$$

Adding inequalities (10) and (11), we get

$$(\lambda^T f)(x,y) + (x^T z) - (y^T \nabla_y(\lambda^T f(x,y))) - (y^T \nabla_{yy}(h^T g(x,y))p) \geqq (\lambda^T f)(u,v) + (v^T w) - (u^T \nabla_x(\lambda^T f(u,v))) - ((u^T \nabla_{xx}(h^T g(u,v))q).$$

Finally, since $x^T z \leqq s(x|D),\ v^T w \leqq s(v|E)$, we obtain

$$(\lambda^T f)(x,y) + s(x|D) - (y^T \nabla_y(\lambda^T f(x,y))) - (y^T \nabla_{yy}(h^T g(x,y))p) \geqq (\lambda^T f)(u,v) + s(v|E) - (u^T \nabla_x(\lambda^T f(u,v))) - (u^T \nabla_{xx}(h^T g(u,v))q),$$

which contradicts inequality (7). Thus the result holds.

Theorem 2 (Strong duality)

Let $(\bar{x},\bar{y},\bar{\lambda},\bar{h},\bar{w},\bar{p})$ be an efficient solution for (WP). Suppose that

(i) $\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})$ is nonsingular,

(ii) the set $\{\nabla_y f_1(\bar{x},\bar{y}), \nabla_y f_2(\bar{x},\bar{y}), ...\nabla_y f_k(\bar{x},\bar{y})\}$ is linearly independent, and

(iii) $\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p} \notin \text{span}\{\nabla_y f_1(\bar{x},\bar{y}), \nabla_y f_2(\bar{x},\bar{y}), ...\nabla_y f_k(\bar{x},\bar{y})\} \setminus \{0\}$

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{h},\bar{p}=0)$ is feasible for $(WD)_{\bar{\lambda}}$ and the objective function values of (WP) and $(WD)_{\bar{\lambda}}$ are equal. Also, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of $(WP)_{\bar{\lambda}}$ and $(WD)_{\bar{\lambda}}$, then $(\bar{x},\bar{y},\bar{\lambda},\bar{h},\bar{p}=0)$ is an efficient solution for $(WD)_{\bar{\lambda}}$.

Proof

Since $(\bar{x},\bar{y},\bar{\lambda},\bar{h},\bar{p})$ is an efficient solution for (WP), by the Fritz John necessary optimality conditions [7], there exist $\bar{\alpha} \in R^k, \bar{\beta} \in R^m, \bar{\delta} \in R, \bar{\xi} \in R^k$ and $\bar{\eta}, \bar{z} \in R^n$ such that the following necessary conditions are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{h},\bar{w},p)$:

$$(\nabla_x f(\bar{x},\bar{y}) + \bar{\gamma} e^T)\bar{\alpha} + [(\nabla_{yx}(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_x((\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p}](\bar{\beta} - (\bar{\alpha}^T e)\bar{y}) - \bar{\eta} = 0, \quad (12)$$

$$\nabla_y f(\bar{x},\bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) + [(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_y((\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p}](\bar{\beta} - (\bar{\alpha}^T e)\bar{y}) - (\bar{\alpha}^T e)\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p} = 0, \quad (13)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T \nabla_y f(\bar{x},\bar{y}) + \bar{\delta} e^T - \bar{\xi} = 0, \quad (14)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T \nabla_h(\nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p}) = 0, \quad (15)$$

$$(\bar{\beta} - (\bar{\alpha}^T e)\bar{y})^T \nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y}) = 0, \quad (16)$$

$$\bar{\beta} \in N_E(\bar{w}), \quad (17)$$

$$\bar{\delta}(\bar{\lambda}^T e - 1) = 0, \quad (18)$$

$$\bar{\lambda}^T \bar{\xi} = 0, \quad (19)$$

$$\bar{x}^T \bar{\eta} = 0, \quad (20)$$

$$\bar{\beta}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) - \bar{w} + \nabla_{yy}(\bar{h}^T g)(\bar{x},\bar{y})\bar{p}] = 0 \quad (21)$$

$$\bar{\gamma} \in D, \bar{\gamma}^T \bar{x} = s(\bar{x}|D), \quad (22)$$

$$(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\xi},\bar{\eta}) \geqq 0, \quad (23)$$

$$(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta},\bar{\xi},\bar{\eta}) \neq 0, \quad (24)$$

As $\bar{\lambda} > 0$, from (19) we conclude that $\bar{\xi} = 0$. By hypothesis (i), equation (16) implies

$$\bar{\beta} = (\bar{\alpha}^T e)\bar{y}. \quad (25)$$

Therefore (14) yields $\bar{\delta} = 0$.

Now suppose $\bar{\alpha} = 0$. Then equation (25) implies $\bar{\beta} = 0$. Also, equation (12) implies that $\bar{\eta} = 0$. Hence, $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\xi}, \bar{\eta}) = 0$, which contradicts (24). Hence, $\bar{\alpha} \neq 0,$ So $\bar{\alpha} \geq 0,$ or

$$\bar{\alpha}^T e > 0. \quad (26)$$

Therefore equations (25) and (26) yield

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e} \geqq 0.$$

Now, from (13) and (25), we have

$$\nabla_y f(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) = (\bar{\alpha}^T e)\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p}. \quad (27)$$

Using the hypothesis (iii), the above relation implies $(\bar{\alpha}^T e)\nabla_{yy}(\bar{h}^T g)(\bar{x}, \bar{y})\bar{p} = 0$ which by hypothesis (i) and (26) yield

$$\bar{p} = 0. \quad (28)$$

Therefore equation (27) gives

$$(\nabla_y f)(\bar{x}, \bar{y})(\bar{\alpha} - (\bar{\alpha}^T e)\bar{\lambda}) = 0.$$

Since the set $\{\nabla_y f_1(\bar{x}, \bar{y}), \nabla_y f_2(\bar{x}, \bar{y}), \ldots\ldots \nabla_y f_k(\bar{x}, \bar{y})\}$ is linearly independent, the above equation implies

$$\bar{\alpha} = (\bar{\alpha}^T e)\bar{\lambda}. \quad (29)$$

Using (25), (26) and (29) in (12), we get

$$(\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \bar{\gamma}) = \eta \geqq 0. \quad (30)$$

Thus $(\bar{x}, \bar{y}, \bar{h}, \bar{z} = \bar{\gamma}, \bar{q} = 0)$ is a feasible solution for the problem $(WD)_{\bar{\lambda}}$.

Now from equation (30),

$$\bar{x}^T(\nabla_x((\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \bar{z}) = \bar{\eta}^T \bar{x} = 0$$

or using (22)

$$\bar{x}^T(\nabla_x((\bar{\lambda}^T f)(\bar{x}, \bar{y})) = -\bar{x}^T \bar{z} = -s(\bar{x} \mid D). \quad (31)$$

Also, as E is a compact convex set in R^m, $\bar{y}^T \bar{w} = s(\bar{y} \mid E).$

Further, from (21), (25), (26) and (28), we obtain

$$\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \bar{y}^T \bar{w} = s(\bar{y} \mid E). \quad (32)$$

Thus, the two objective function values are equal. Using weak duality it can be easily shown $(\bar{x}, \bar{y}, \bar{h}, \bar{z}, \bar{q} = 0)$ is an efficient solution of $(WD)_{\bar{\lambda}}$.

Theorem 3 (Converse duality)

Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{h}, \bar{z}, \bar{q})$ be an efficient solution for $(WD).$ Suppose that

(i) $\nabla_{xx}(\bar{h}^T g)(\bar{u}, \bar{v})$ be nonsingular,

(ii) the set $\{\nabla_x f_1(\bar{u}, \bar{v}), \nabla_x f_2(\bar{u}, \bar{v}), \ldots\ldots, \nabla_x f_k(\bar{u}, \bar{v})\}$ is linearly independent, and

(iii) $\nabla_{xx}(\bar{h}^T g)(\bar{u}, \bar{v})\bar{q} \notin \text{span}\{\nabla_x f_1(\bar{u}, \bar{v}), \nabla_x f_2(\bar{u}, \bar{v}), \ldots. \ldots\ldots\ldots\ldots, \nabla_x f_k(\bar{u}, \bar{v})\} \setminus \{0\}.$

Then $(\bar{u}, \bar{v}, \bar{h}, \bar{z}, \bar{p} = 0)$ is feasible for $(WP)_{\bar{\lambda}}$ and the objective function values of $(WP)_{\bar{\lambda}}$ and (WD) are equal. Furthermore, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of $(WP)_{\bar{\lambda}}$ and $(WD)_{\bar{\lambda}}$ then $(\bar{u}, \bar{v}, \bar{h}, \bar{w}, \bar{q} = 0)$ is an efficient solution for $(WP)_{\bar{\lambda}}$.

Proof

Follows on the lines of Theorem 2.

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