Journal of Statistics and Mathematics
Journal of Statistics and Mathematics
ISSN: 0976-8807 & E-ISSN: 0976-8815, Volume 2, Issue 2, 2011, PP-37-39
Online available at : http://www.bioinfo.in/contents.php?id=85

EXACT SOLUTION FOR THE INVERSE PROBLEM OF FINDING A SOURCE TERM IN A PARABOLIC EQUATION

JAFARI M.A. AND AMINATAEI A.*
Department of Mathematics, K. N. Toosi University of Technology, P. O. Box: 16315-1618, Tehran, Iran
*Corresponding author. E-mail: ataei@kntu.ac.ir

Received: August 30, 2011; Accepted: September 20, 2011

Abstract- In this paper, the homotopy perturbation method is applied to solve inverse problem of finding an unknown source term in a parabolic equation. In this method, the solution is found in the form of a convergent series. The terms of the series can be computed easily. To illustrate the method, three examples are presented. The results show the simplicity and efficiency of the method.
Key words -Inverse problem; Parabolic equation; Unknown source term; Homotopy perturbation method.

Introduction

Inverse problems have many applications in various fields of science and engineering such as mechanic and chemistry engineering. An efficient numerical method for solving inverse problems has been discussed by several authors (for further see [1,3,9] and references therein). Recently homotopy perturbation method (HPM) has been applied with great success to obtain approximate solutions for a large variety of linear and nonlinear problems in ordinary differential equations (ODEs) [6], partial differential equations (PDEs) [7], and integral equations [10]. In addition, some modifications for HPM have been suggested [2,4,8]. The present paper is devoted to apply HPM for solving the inverse problem of identifying an unknown source term in a parabolic equation as

$$\rho(\bar{x},t)u_t - \Delta u = F(\bar{x},t), (\bar{x},t) \in Q \equiv \Omega \times [0,T], \quad (1)$$

where $F(\bar{x},t) = f(\bar{x})g(\bar{x},t) + h(\bar{x},t)$, $g(\bar{x},t)$ and $h(\bar{x},t)$ are two known functions, and $f(\bar{x})$ is an unknown function. Furthermore, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. The initial and boundary conditions are given by

$$\begin{cases} u(\bar{x},0) = u_0(\bar{x}), \bar{x} \in \Omega, \\ u(\bar{x},0) = u_0(\bar{x}), \bar{x} \in \Omega. \end{cases} \quad (2)$$

In addition, we have the following over determination condition

$$u(\bar{x},T) = r(\bar{x}), \bar{x} \in \Omega. \quad (3)$$

The problem (1)-(3) has successfully modeled many phenomena such as air pollution phenomena. Setting $t = T$ in Eq. (1) yields

$$\rho(\bar{x},T)u_t(\bar{x},T) - \Delta u(\bar{x},T) = f(\bar{x})g(\bar{x},T) + h(\bar{x},T). \quad (4)$$

Over determination condition (3) simplifies (4) as

$$\rho(\bar{x},T)u_t(\bar{x},T) - \Delta r(\bar{x}) = f(\bar{x})g(\bar{x},T) + h(\bar{x},T). \quad (5)$$

Consequently, if $g(\bar{x},T) \neq 0$ we obtain

$$f(\bar{x}) = \frac{\rho(\bar{x},T)u_t(\bar{x},T) - \Delta r(\bar{x}) - h(\bar{x},T)}{g(\bar{x},T)}. \quad (6)$$

Substituting (6) into Eq. (1) yields

$$\Delta u(\bar{x},t) - \rho(\bar{x},t)u_t(\bar{x},t) + \frac{\rho(\bar{x},T)g(\bar{x},t)}{g(\bar{x},T)}u_t(\bar{x},T) = \frac{h(\bar{x},T)g(\bar{x},t)}{g(\bar{x},T)} + \frac{g(\bar{x},t)\Delta r(\bar{x})}{g(\bar{x},T)} - h(\bar{x},t). \quad (7)$$

In this paper HPM is applied to solve (7) with aforesaid initial and boundary conditions in (2).

This paper is organized as follows. In section 2, for convenience of the reader, a short review of HPM is stated. In section 3, HPM is applied to solve some examples of problem (1)-(3). Finally, a short conclusion is given in section 4.

Homotopy Perturbation Method (HPM)

J.H. He presented a homotopy perturbation technique based on the introduction of a homotopy and an artificial parameter for the solution of algebraic and ODEs [5]. To explain HPM, we consider the following nonlinear differential equation

$$A(v) - f(r) = 0, r \in \Omega, \quad (8)$$

with the boundary conditions

$$B(v, \frac{\partial v}{\partial n}) = 0, r \in \Gamma, \quad (9)$$

where A is a differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can be divided into parts L and N, where L is a linear operator and N is a nonlinear operator. Therefore, equation (8) can be rewritten as

$$L(v) + N(v) - f(r) = 0. \quad (10)$$

Now, a homotopy $v(r,p): \Omega \times [0,1] \to R$ is constructed as follows:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = L(v) - (1-p)L(u_0) + p[N(v) - f(r)] = 0, r \in \Omega, \quad (11)$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation which satisfies the boundary conditions. Obviously, we have

$$\begin{cases} H(v,0) = L(v) - L(u_0) = 0, \\ H(v,1) = A(v) - f(r) = 0. \end{cases} \quad (12)$$

Changing the p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $v(r)$. We assume that the solution of equation (11) can be written as a power series in the following equation:

$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots = \sum_{i=0}^{\infty} p^i v_i$. (13)

By setting (13) in (11) and equating the terms by the same power in p, a successive procedure to determine v_i is obtained. As a result, to solve (7) in the one-dimension, the following homotopy is considered

$$H(v,p) = \frac{\partial^2 v}{\partial x^2} - (1-p)\frac{\partial^2 u_0}{\partial x^2} + p\left[-\rho(x,t)\frac{\partial v}{\partial t} + \frac{\rho(x,T)g(x,t)}{g(x,T)}\left(\frac{\partial v}{\partial t}|_{t=1}\right) - \frac{h(x,T)g(x,t)}{g(x,T)} - \frac{g(x,t)\Delta r(x)}{g(x,T)} + h(x,t)\right]. \quad (14)$$

Generally, v_i for $i = 0,1,2,\ldots$ are obtained from the following recursive relations:

$$\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \to v_0 = u_0,$$

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_0}{\partial x^2} - \rho(x,t)\frac{\partial v_0}{\partial t} + \frac{\rho(x,T)g(x,t)}{g(x,T)}\left(\frac{\partial v_0}{\partial t}|_{t=1}\right)$$

$$-\frac{h(x,T)g(x,t)}{g(x,T)} - \frac{g(x,t)\Delta r(x)}{g(x,T)} + h(x,t) = 0,$$

$$\frac{\partial^2 v_2}{\partial x^2} - \rho(x,t)\frac{\partial v_1}{\partial t} + \frac{\rho(x,T)g(x,t)}{g(x,T)}\left(\frac{\partial v_1}{\partial t}|_{t=1}\right) = 0,$$

$$\frac{\partial^2 v_n}{\partial x^2} - \rho(x,t)\frac{\partial v_{n-1}}{\partial t} + \frac{\rho(x,T)g(x,t)}{g(x,T)}\left(\frac{\partial v_{n-1}}{\partial t}|_{t=1}\right) = 0,$$

$$\vdots \qquad \vdots$$

and initial conditions $v_i(x,0) = \frac{\partial}{\partial x}v_i(x,0) = 0$ for $i \in \mathbb{N}$.

Illustrative examples

In order to illustrate the method, we consider three examples [3,9].

Example 1

Consider $\rho(x,t) = 1, g(x,t) = e^t, h(x,t) = 0, u(x,0) = e^{2x}, u(0,t) = e^t, u(1,t) = e^{t+2}$, and $r(x) = e^{2x+1}$ on the domain $Q \equiv \{(x,t)|(x,t) \in [0,1] \times [0,1]\}$. Hence, by using (7) we obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + \frac{e^t}{e}u_t(x,T) = 4e^{2x+t}.$$

We set $u_0 = u(x,0) = e^{2x}$. According to the HPM that is defined in (14), we have

$$H(v,p) = \frac{\partial^2}{\partial x^2}v - (1-p)\frac{\partial^2 u_0}{\partial x^2} + p\left[-\frac{\partial}{\partial t}v + \frac{e^t}{e}\left(\frac{\partial}{\partial t}v|_{t=1}\right) - 4e^{2x+t}\right] = 0,$$

where v is defined in (13). By equating the coefficients of p to zero, we obtain

coefficient of p^0: $\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \Rightarrow v_0 = u_0 = e^{2x}$,

coefficient of p^1: $\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial v_0}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_0}{\partial t}|_{t=1}\right) - 4e^{2x+t} = 0 \Rightarrow v_1 = e^{2x}(e^t - 1)$,

coefficient of p^2: $\frac{\partial^2 v_2}{\partial x^2} - \frac{\partial v_1}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_1}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_2 = 0$,

$\vdots$

coefficient of p^n: $\frac{\partial^n v_n}{\partial x^2} - \frac{\partial v_{n-1}}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_{n-1}}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_n = 0$.

Therefore, we obtain $u(x,t) = e^{2x+t}$, which is the exact solution of the problem. By using (6), we obtain $f(x) = -3e^{2x}$.

Example 2

Consider $\rho(x,t) = 1, g(x,t) = e^t, h(x,t) = 0, u(x,0) = \sin x, u(0,t) = 0, u(1,t) = \sin(1)e^t$, and $r(x) = e\sin x$ on the domain $Q \equiv \{(x,t)|(x,t) \in [0,1] \times [0,1]\}$. Hence, by using (7) we obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + \frac{e^t}{e}u_t(x,T) = -e^t\sin x.$$

We set $u_0 = u(x,0) = \sin x$. According to the HPM that is defined in (14), we have

$$H(v,p) = \frac{\partial^2}{\partial x^2}v - (1-p)\frac{\partial^2 u_0}{\partial x^2} + p\left[-\frac{\partial}{\partial t}v + \frac{e^t}{e}\left(\frac{\partial}{\partial t}v|_{t=1}\right) + e^t\sin x\right] = 0,$$

where v is defined in (13). By equating the coefficients of p to zero, we obtain

coefficient of p^0: $\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \Rightarrow v_0 = u_0 = \sin x$,

coefficient of p^1: $\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial v_0}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_0}{\partial t}|_{t=1}\right) + e^t\sin x = 0 \Rightarrow v_1 = -\sin x + e^t\sin x$,

coefficient of p^2: $\frac{\partial^2 v_2}{\partial x^2} - \frac{\partial v_1}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_1}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_2 = 0$,

$\vdots$

coefficient of p^n: $\frac{\partial^n v_2}{\partial x^2} - \frac{\partial v_{n-1}}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_{n-1}}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_n = 0$.

Therefore, we obtain $u(x,t) = e^t\sin x$, which is the exact solution of the problem. By using (6), we obtain $f(x) = 2\sin x$.

Example 3

Consider $\rho(x,t) = 1, g(x,t) = e^t, h(x,t) = 0, u(x,0) = \cosh x, u(0,t) = e^t, u(1,t)e^t\cosh 1$, and $r(x) = e\cosh x$ on the domain $Q \equiv \{(x,t)|(x,t) \in [0,1] \times [0,1]\}$. Hence, by using (7) we obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + \frac{e^t}{e}u_t(x,T) = e^t\cosh x.$$

We set $u_0 = u(x,0) = \cosh x$. According to the HPM that is defined in (14), we have

$$H(v,p) = \frac{\partial^2}{\partial x^2}v - (1-p)\frac{\partial^2 u_0}{\partial x^2} + p\left[-\frac{\partial}{\partial t}v + \frac{e^t}{e}\left(\frac{\partial}{\partial t}v|_{t=1}\right) - e^t\cosh x\right] = 0,$$

where v is defined in (13). By equating the coefficients of p to zero, we obtain

coefficient of p^0: $\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \Rightarrow v_0 = u_0 = \cosh x$,

coefficient of $p^1: \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial v_0}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_0}{\partial t}|_{t=1}\right) - e^t \cosh x = 0 \Rightarrow v_1 = \cosh x(e^t - 1)$,
coefficient of $p^2: \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial v_1}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_1}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_2 = 0$,

$\vdots$

coefficient of $p^n: \frac{\partial^n v_2}{\partial x^2} - \frac{\partial v_{n-1}}{\partial t} + \frac{e^t}{e}\left(\frac{\partial v_{n-1}}{\partial t}|_{t=1}\right) = 0 \Rightarrow v_n = 0$.

Therefore, we obtain $u(x,t) = e^t \cosh x$, which is the exact solution of the problem. By using (6), we obtain $f(x) = 0$.

Conclusion

In this paper, HPM is applied for solving the inverse problem of identifying an unknown source term in a parabolic equation. According to the solutions that we have obtained, we infer the HPM is a powerful tool for solving this kind of inverse problem. Additionally, this method avoids the round-off errors.

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