Journal of Statistics and Mathematics
ISSN: 0976-8807 & E-ISSN: 0976-8815, Volume 2, Issue 1, 2011, PP-07-14
Online available at : http://www.bioinfo.in/contents.php?id=85

GAMES IN LOGIC, LOGIC IN GAMES, AND META GAMES

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Received: May 09, 2011; Accepted: June 19, 2011

Abstract- This is a survey on the relationship between logic and games. What do games have to save about logic, and conversely what does logic have to say about games? Johan Van Benthem in his lengthy manuscript Logic In Games[1] sets forth an axiomatization of game equivalence and asks two main questions. The first is whether the axioms are complete for the semantic notion of game equivalence. Van Benthem also cautions that it is important to distinguish that games are "dynamic" activities, and that the meaning of a game is not fully captured by the assertion player has a winning strategy in it, and hence the second question is what constitutes this "dynamic aspect".
In this survey project, I will briefly discuss the difference between using games to determine results about logic and using logic to determine results about games. I then will discuss two responses in the affirmative to the first question by Van Benthem about the axiomatization of game logic with regard to logic in games. One is by Goranko which employs translations into modal logic to obtain the completeness result; the second is by Venema which uses a more general approach to show that game algebras and board algebras are isomorphic. I will also offer what seems to be a novel approach in responding to Van Benthem's second question by suggesting that games are not fully captured by understanding whether a player has a winning strategy or not because games involve a dynamic action between intelligent agents who are trying to out think each other. In order to represent this dynamic process mathematically I propose that one must classify strategies themselves, and I will suggest ways of classifying strategies in the context of modal logic.

Introduction

There is a close connection between logic and games in the sense that certain theorems about logic have game theoretic counterparts. For example, there is a connection between truth in a model and wining strategy in games. Henkin style completeness results can be viewed in terms of games. Games are employed in descriptive complexity and finite model theory with EhrenFreuct- Fraisse games and pebble games. Logic, however, can also be used as a general meta-theory of games. What is a game? When are two games equivalent? What are game invariances? Along these lines, we can examine game theory from the standpoint of an algebra of games, operations on games, and an axiomatization of game theory with respect to the standard operations.

Games in Logic vs. Logic in Games:

Games in Logic:

Standard Connection: Semantic Evaluation Games

Propositional Logic: Given some fixed evaluation τ for propositional atoms, a game is defined for each formula φ between V(verifier) and F(falsifier).

Negation changes player roles. Disjunction allows V to chose. Conjunction allows F(falsifier) to chose.
Winning or losing occurs at an atom, where if it is true according to τ, V wins. If it is false according to τ, F wins.
Easy Result: Formula φ is true under τ iff Verifier has a winning strategy in the game for φ.

1-st Order Logic: The rules are extended to first order logic where given $\exists x\varphi$, V picks an object d, and play continues with respect to $\varphi[x/d]$. Given $\forall x\varphi$, F chooses the object. Atomic formulas are decided by the relations in the model.
By induction on formula complexity, it is not hard to show that
φ is true in M iff Verifier has a winning strategy for the evaluation game played in M.
Some Standard Results in Descriptive Complexity using Games:
Theorem (Fraise-Ehrenfeuct):

A and B satisfy the same first order sentences of quantifier rank r iff the Duplicator wins the r-move E-F game on A and B

[1] *Van Benthem, J., Logic in Games, Lecture notes, ILLC, University of Amsterdam, 2000*

Definability and E-F Games: Given a Boolean query Q on a class of structures C closed under isomorphisms, Q is first order definable on C iff there is an r such that for all A, B in C if A |= Q and the Duplicator wins the r move E-F game on A and B then B|=Q

Methodology for First-Order Definability: If for every r, there exist structures A and B such that A |= Q and B |≠ Q and the Duplicator wins the r-move E-F game on A, B then Q is not first order definable

Theorem (Barwise-Immerman)[2]: A and B satisfy the same $L^k_{\infty w}$ sentences iff the Duplicator wins the k-pebble game on A and B

Definability and Pebble Games[3]: Given a Boolean query Q on a class of structures C closed under isomorphisms, Q is $L^w_{\infty w}$- definable on C iff there exists a k such that for all A, B in C if A |= Q and the Duplicator wins the k-pebble game on A and B then B|=Q

A standard result for the methodology of $L^w_{\infty w}$-definability : A query is not $L^w_{\infty w}$- definable if there are for every k, structures A_k and B_k such A_k |= Q and B_k |= Q but the Duplicator wins the k-pebble game on A_k and B_k.

Proof of Completeness Theorem in 1-st Order Logic via Games:

The Game G_φ is played in which II claims there is a model and I challenges this claim.

Define the rank of a formula by $p(\varphi) = \text{length}(\varphi) + \max\{ i: x_i \text{ occurs in } \varphi\}$

$\varphi = \psi(x_0,.., x_n)$. In each round of the game, I plays a formula and II responds by playing either 0 or 1.

Rules:

1) Each $\varepsilon_i = 0,1$
2) Each φ_i is a formula with $| \varphi_i | < i + 2$
3) If $\varphi_i = \exists x_j\ \psi(x_1,..x_n)$ and $\varepsilon_i = 1$ then II must play 1 in response to $\psi(x_1,..x_{j-1},x_{|\varphi| + i + 1}, x_{j+1},\ldots,x_n)$
4) If $\varphi_i = \varphi$, $\varepsilon_i = 1$
5) If $\varphi_i = \varphi_k \wedge \varphi_j$ then $\varepsilon_i = 1$ iff $\varepsilon_k = 1$ and $\varepsilon_j = 1$
6) If $\varphi_i = \neg\varphi_k$ then $\varepsilon_i = 1$ iff $\varepsilon_k = 0$

The first player to play against the rules loses

Lemma: Suppose II wins the game G_φ then there exists a structure M and $a_1,..,a_n$ such that M |= $\varphi[a_1,..,a_n]$

(induction on formula complexity)

Gale Stewart Determinacy for Open games[4]

$A \subset X^w = \{f: w \text{ -> } X \}$ Associate to A a game G_A .

Rules: $a_i, b_i \in X$, $f(2^i) = a_i$, $f(2^{i+1}) = b_i$, I wins if $f \in A$

G_A is determined if there is a function $\tau: X^{<w} \to X$ which is a winning strategy for I or II

Proof of Determinacy for open games: Assume A is open so that for all $f \in A$ there exists $n \in w$, s.t. $\{g \in X^w| g|n = f|n\} \subset A$ Define a sequence of subsets of $X^{<w}$ by transfinite induction as follows

$A_0 = \{t \in X^{<w} \mid \{f: f|_{l(t)} = t\} \subset A\}$ Note: these are positions at which I has won

Suppose A_α is defined then $A_{\alpha+1}$ is the set of all $t \in X^{<w}$ such that

if t has even length then $t \wedge a \in A_\alpha$ for some $a \in X$, if t has odd length then $t \wedge b \in A_\alpha$ for all $b \in X$

if β is a limit stage then $A_\beta = \cup_{\alpha<\beta} A_\alpha$ Note: there must exist α such that $A_\alpha = A_{\alpha+1}$

Let $A\infty = \cup A_\alpha$

Two cases:

1) $\varnothing \in A^*$ (corresponds to I wins G_A)
2) $\varnothing \notin A^*$ (corresponds to II wins G_A)

Let $\rho: X^{<w} \to \text{Ordinals} \cup \{\infty\}$

be the function such that $\rho(t)$ = least α such that $t \in A_{\alpha+1}$

Lemma) Let $t \in X^{<w}$, Suppose $p(t) < \infty$

then

1) if t has even length then $\exists a \in X$ with $p(t \wedge a) < p(t)$ or $p(t) = 0$;

if t has odd length then $p(t) = 0$ or $p(t \wedge b) < p(t)$ for all $b \in X$

2) if $p(t) = \infty$ then if t has even length $p(t \wedge a) = \infty$ for all $a \in X$,

if t has odd length then $p(t \wedge b) = \infty$ for some b

If $\rho(\varnothing) < \infty$ then I's winning strategy is to play to reduce ρ

If $\rho(\varnothing) = \infty$ then II's winning strategy is to play to preserve this

Proof: Assume that player I doesn't have a winning strategy, then there are moves by II so that no matter what player I does, player I can't win. WTS this is a winning strategy for II. Suppose not, then since A is open , in this run of the game, f, there is some finite stage $f\lceil n$, such that all extensions are in A, so $\rho(\varnothing) < \infty$, and I has a winning strategy at this cut

Logic in Games:

Game Algebra:

Algebra for Game Equivalence(Goranko)

atomic games $\{g_a\}_{a \in A}$

Game operations: $\vee$, ^{d} , $^\circ$

Definition: $G \wedge H = (G^d \vee H^d)^d$

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[2] *Kolaitis (2002)*

[3] *Kolaitis (2002)*

[4] *Hodges (1993) 116*

$\vee$ refers to choice of first player (as to which game is being played)
^{d} refers to swapping the role of the two players
$\circ$ refers to sequential game composition

Right distribution fails in game algebra:
$G_1 \circ (G_2 \vee G_3) \neq (G_1 \circ G_2) \vee (G_1 \circ G_3)$
The reason for this is essentially that $[\forall x(\varphi(x) \vee \psi(x)) \leftrightarrow (\forall x\ \varphi(x) \vee \forall x\ \psi(x))]$ is not valid in first order logic. A player may have winning strategy to end game 1 in a position where she can either win game2 or game 3, that does not mean that she can always end in position where she can win game 2 or always end in a position where she can win game 3.

Axioms for Algebra of Games:
1) $G = G^{dd}$
2) $\vee$ is commutative, associative, and indempotent
3) $G \vee (G \wedge H) = G$
4) $G \vee (H \wedge M) = (G \vee H) \wedge (G \vee M)$
5) $\circ$ is associative
6) $(G \circ H)^d = G^d \circ H^d$
7) $(G_2 \vee G_3) \circ G_1 = (G_2 \circ G_1) \vee (G_3 \circ G_1)$
8) $G_1 \circ (G_2 \vee G_3) \geq (G_1 \circ G_2)$
9) $G = G \circ \iota = \iota \circ G,\ \iota = \iota^d$

Models for Game Logic are Game Boards, $< S, \{\rho^i_a\}_{a \in A,\ i = 1,2} >$ where S is a set of states and ρ^i_a is an atomic forcing relations satisfying forcing conditions, $\rho^i_a \subseteq S \times P(S)$, of upwards monotonicity, and consistency of powers
Forcing relations are extended to forcing relations for all game terms in the natural recursive way.
$sp^1_{G^d}X$ iff $sp^2_G X$
$sp^2_{G^d}X$ iff $sp^1_G X$
$sp^i_{G \vee H}$ iff $sp^i_G X$ or $sp^i_H X$
$sp^i_{G \circ H} X$ iff there exists Z such that $sp^i_G Z$ and $zp^i_H X$ for each $z \in Z$

The meaning of $sp^i_G X$ is that from state s, player i has a strategy to play the game G so that if an outcome state is attained it is in X

$\geq$ is a constant symbol of the language, and it is to be interpreted as game inclusion in any model, that is given any game board, B, B $\models H \geq G$ if $G \subseteq_i H$ for each i on B, where
$G \subseteq_i H$ means that $p^i_G \subseteq p^i_H$

Van Benthem's first question in his manuscript is whether these axioms constitute a complete axiomatization for the semantic notion of equivalence. That it does was proved first by Goranko[5].

[5] *Goranko, V.F., 'The basic algebra of game equivalences', Studia Logica, 75:221-238, 2003*

(Goranko's article can be somewhat confusing because it often convolutes syntactic and semantic notions)
Goranko's Completeness result: Every valid game term identity of the game algebra can be derived from GA in the standard equational logic[6]
Goranko's result relies heavily on a translation of game identities into formulas of modal logic which preserves validities:

Definition of Canonical Game Terms
ι is a canonical term
Let $\{G_{ik} | k \in K_i,\ i \in I\}$be finite nonempty family of canonical terms and $\{g_{ik} | k \in K_i,\ i \in I\}$ be a family of literals such that g_{ik} can be idle only if G_{ik} is idle, then $\vee_{i \in I} \wedge_{k \in Ki}\, g_{ik} \circ G_{ik}$ is a canonical term

Translation into modal logic:

V = set of atomic variables = $\{p_a\}$

Let $\varphi(q/\psi)$ be the formula resulting from substituting ψ for all occurrences of q (note that ψ may be a formula of modal logic and not necessarily just a variable)

q is an auxiliary variable

$M(\iota) = q$
$M(g_a) = \Diamond\Box (p_a \rightarrow q)$
$M(g_a{}^d) = \Box\Diamond(p_a \wedge q) = \neg (M(g_a))(\neg q/q)$
$M(G \vee H) = M(G) \vee M(H)$
$M(G \circ H) = M(G)[M(H)/q]$

A key observation is that in translating the dual of a game, it is not just the negation of the formula for the complement of winning states for player one is being included(this is essentially the role of substituting $\neg q/q$. I wins when it is possible to make a move so that no mater what II does an outcome state where I wins is reached. II wins if no matter what I does it is always possible to make a move so that the outcome state is in the complement of winning states for I.

Theorem[7]: (Preservation of validity)For any game terms G,H, if $H \geq G$ is valid on all determined game boards then
$\models m(G) \rightarrow m(H)$

Pf) By contraposition, suppose M, u does not satisfy $m(G) \rightarrow m(H)$,

[6] *Goranko, V.F., 'The basic algebra of game equivalences', Studia Logica, 75:226, 2003*
[7] *Goranko, V.F., 'The basic algebra of game equivalences', Studia Logica, 75:229, 2003*

define the game board B as follows: For every $X \subset S$ and $s \in S$:

sp^1_aX iff $M,s \models m(g_a)$ under an evaluation that maps q to X
sp^2_aX iff $M,s \models m^d(g_a)$ under an evaluation that maps q to X

Now,
by structural induction on the complexity of terms,
sp^1_DX iff $M,s \models m(g_a)$ under an evaluation that maps q to X
sp^2_DX iff $M,s \models m^d(g_a)$ under an evaluation that maps q to X
so up^1_GX but $\neg\ up^1_HX$

Definition: Let $F_* = \langle S_*, R_* \rangle$ where $S_* = \{*, y, z\}$, $R_* = \{(*,y), (y,z), (z,z)\}$then the Kripke model $M_+ = \langle S_*, R_*, V_+ \rangle$ satisfies all m(G) at its root, *, and $V_+(q) = \{*, z\}$
while $M\_ = \langle S, R_*, V_- \rangle$ $V_-(q) = \varnothing$ and $V_-(p_a) = \{z\}$ for all a, and falsifies all M(G) at *
The reason for this for example in the case of $\Diamond\Box(p_a \to q)$ and M_+ is that one can move from * to y and then necessarily the only move is to z where q is evaluated as true so the conditional holds

The main difficult technical lemma that Goranko employs is as follows:

TFAE:
1) it is not the case that $G \leq H$
2) there exists a disjunct $\wedge_{k \in Ki}\, g_{ik} \circ G_{ik}$ in G such that every disjunct in H contains some conjunct $h_{jmi} \circ H_{jmi}$ not including any of the conjuncts $g_{ik} \circ G_{ik}$ for any k
3) There is a finite Kripke model M and a state s such that $M, s \models m(G)$ but that M, s does not satisfy m(H) and s has no predecessors in M

Proof of the technical lemma in Goranko[8]:

1) $\to$ 2) Assume it is not the case that $G \leq H$ then wlog let $G \not\subset_1 H$, then there is a state s and the choice of player one as some $\wedge_{k \in Ki}\, g_{ik} \circ G_{ik}$ such that every conjunct $g_{ik} \circ G_{ik}$ enables him to reach some outcome X from s that cannot be forced on H, and hence every disjunct in H must contain a conjunct which lacks the power for player 1 to force an outcome in X, and hence none of the terms $h_{jmi} \circ H_{jmi}$ can include any of the $g_{ik} \circ G_{ik}$
2) $\to$ 3) A Kripke model will be constructed which will satisfy all $\{m(g_{ik} \circ G_{ik})\}$ and none of the $m(h_{jmi} \circ H_{jmi})$, M will be rooted at some state s with no predecessors

The set of all terms must break up into the following subsets:
$T_A = \{t_\alpha \circ D_\alpha \mid \alpha \in A\}$ whose translations must be true at s
$T_B = \{t^d_\beta \circ D_\alpha \mid \beta \in B\}$ whose translations must be true at s
$T_\Gamma = \{t_\gamma \circ D_\gamma \mid \gamma \in \Gamma\}$ whose translations must be false at s
$T_B = \{t^d_\delta \circ D_\delta \mid \delta \in \Delta\}$ whose translations must be false at s

so any suitable Kripke model would have to satisfy the following sets of formulas
$F_A = \{\Diamond\Box(p_\alpha \to m(D_\alpha)) \mid \alpha \in A\}$
$F_B = \{\Box\Diamond(p_\beta \wedge m(D_\beta)) \mid \beta \in B\}$
$F_\Gamma = \{\Box\Diamond(p_\gamma \wedge \neg m(D_\gamma)) \mid \gamma \in \Gamma\}$
$F_\Delta = \{\Diamond\Box(p_\delta \to \neg m(D_\delta)) \mid \delta \in \Delta\}$

notice that for $F_\Delta = \{\Diamond\Box(p_\delta \to \neg m(D_\delta)) \mid \delta \in \Delta\}$ $m(t^d_\delta \circ D_\delta) = m(t^d_\delta)[m(D_\delta)/q]$ and $m(t^d_\delta) = \neg(\Diamond\Box(p_\delta \to \neg q))$ so $m(t^d_\delta)[m(D_\delta)/q] = \Diamond\Box(p_\delta \to \neg m(D_\delta))$, the others are similar

Constructing the model:
M = (W,R,V)
$W = \{s\} \cup (A \cup \Delta) \cup (((A \cup \Delta) \times (B \cup \Gamma)) \cup W'$

The index sets form the skeleton of the model and W' will be submodels which will be grafted onto the skeleton

$R = \{(s,x) \mid x \in A \cup \Delta\} \cup \{(x, (x,y)) \mid x \in A \cup \Delta, y \in B \cup \Gamma\} \cup R'$
R' will be the union of the inherited relations from the grafted sub models

Induction on term complexity: IH:
There exists a disjunct $\wedge_{k \in Ki}\, g_{ik} \circ G_{ik}$ in G such that every disjunct in H contains some conjunct $h_{jmi} \circ H_{jmi}$ not including any of the conjuncts $g_{ik} \circ G_{ik}$ for any k
Then there is a finite Kripke model M and a state s such that $M,s \models m(G)$ but that M,s does not satisfy m(H) and s has no predecessors in M

1) Every state (α,β) for $\alpha \in A$, $\beta \in B$ must satisfy $p_\alpha \to m((D_\alpha))$ and $(p_\beta \wedge m(D_\beta))$ so p_β is set true at α_β and graft M_+ at (α,β)
2) Every state (α,γ), $\alpha \in A$, $\gamma \in \Gamma$ must satisfy $p_\alpha \to m(D_\alpha)$ and $p_\gamma \wedge \neg m(D_\gamma)$ If $\alpha \neq \gamma$ we set p_α false and p_γ true at (α,γ) and graft M_- at (α,γ); if $h_\alpha = h_\gamma$ then not $D_\alpha \leq D_\beta$ so by IH, there is a model $M_{\alpha\gamma}$ rooted at some u such that $M_{\alpha\gamma}, u \models m(D_\alpha)$ but does not satisfy $m(D_\gamma)$ so set p_α true and graft $M_{\alpha\gamma}$ at (α,γ)

[8] *Goranko, V.F., 'The basic algebra of game equivalences', Studia Logica, 75:231-232, 2003*

3) Every state (β,δ) , must satisfy $p_\delta \rightarrow \neg m(D_\delta)$ and $p_\beta \wedge m(D_\beta)$ If $\beta \neq \delta$ we set p_δ false and p_β true at (β,δ) and graft M_+ at (β,δ); if $h_\beta = h_\delta$ then not $D_\beta \leq D_\delta$ so by IH, there is a model $M_{\beta\delta}$ rooted at some u such that $M_{\beta\delta}, u \models m(D_\beta)$ but does not satisfy $m(D_\delta)$ so set p_β true and graft $M_{\beta\delta}$ at (β,δ)

4) Every state (γ,δ) must satisfy $p_\delta \rightarrow \neg m(D_\delta)$ and $p_\gamma \wedge \neg m(D_\gamma)$, so set p_γ true, p_δ false and graft M_-

Venema's more general and algebraic approach[9] *to Van Benthem's first question on game axiomatics*:

Venema defines *game algebras* and *board algebras*. Game algebras satisfy the Van Benthem Axioms. Board algebras are defined via outcome relations. An outcome relation being a relation positions and sets of positions to capture that a player in a certain position has a strategy to force play into a certain set of positions. Venema then shows that every game algebra is isomorphic to a board algebra, analogous to the Stone Representation theorem. Venema's approach seems slightly more sophisticated.

Definition: A game algebra, $(G,\vee, \wedge,d,^\circ)$ is a model satisfying Van Benthem's axioms
Definition: For Outcome relation R^i_g p R^i_g T holds if in position p, i can force that the outcome of the game g will be a position in T
Monotonicity: If p R^i_g T and $T \subseteq U$ then p R^i_g U
Consistency: If p R^i_g T then not I p R^{1-i}_g (B - T) [B = set of all positions]

Outcome Relations for games defined through game operations:
Not: g $\vee_i$ h is the game in which the first move is that player i chooses to play g or h
1) p $R^i_{g \vee i\, h}$ T iff p R^i_g T or If p R^i_h T
2) p $R^i_{g \vee 1\text{-}i\, h}$ T iff p R^i_g T and If p R^i_h T
3) p $R^i_{\text{-}g}$ T iff If p $R^{1\text{-}i}_g$ T
4) p $R^i_{g \circ h}$ T iff p R^i_g U for some U such that for all u in U, u R^i_g T

Board Algebras:
Definition: O(B) = P(B x P(B)) denotes the set of outcome relations on B
O_m(B) = set of monotone outcome relations
G(B) = set of all pairs of outcome relations
G_m(B) = set of all pairs of outcome relations
G_{mc}(B) = set of all consistent pairs of monotone outcome relations

Definition: Binary operation $\circ$ on outcome relations
R $\circ$ S = {(p,T) | p R U for some U such that u S T for all u in U}

[9] Yde Venema, *Representation of Game Algebras,* Studia Logica 75:239-256, 2003

Operations on G(B):
1) $(R_1, R_2) \cup (S_1, S_2) = (R_1 \cup S_1, R_2 \cap S_2)$

Note: the reason for this definition is that the second outcome relation in the ordered pair of outcome relations is referring to the ability of what player 2 can force so that the second outcome relation must be weakened if the first is strengthened
2) $(R_1, R_2) \cap (S_1, S_2) = (R_1 \cap S_1, R_2 \cup S_2)$
3) $(R_1, R_2)^d = (R_2, R_1)$
4) (R_1, R_2) □ $(S_1, S_2) = (R_1 \circ S_1, R_2 \circ S_2)$

An outcome algebra of the form (A, $\cup$, $\cap$, d, □) where $A \subseteq G_{mc}(B)$ is a *board algebra*
Theorem(Venema) Every game algebra is isomorphic to a board algebra[10]
Sketch: The lattice reduct $(G, \wedge, \vee)$ is any algebra satisfying
$x \vee x = x$
$x \vee y = y \vee x$
$x \vee (y \vee z) = (x \vee y) \vee z$
$x \vee (x \wedge y) = x$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

The first five of Van Benthem's axioms and such an algebra is isomorphic to a set lattice over the collection of the prime filters of G, denoted by B_G. Basically, something similar to B_G will serve as the underlying set of the board algebra representing G

Definition: A deMorgan reduct $(G,\vee, \wedge,d)$ of a game algebra satisfies the above five axioms and also

$x^{dd} = x$
$(x \vee y)^d = x^d \wedge y^d$

Definition: A module over game algebra G is an algebra $\mathrm{M} = (M, \vee, \wedge, d, \circ_g)$ g ε G such that $(M, \vee, \wedge, d,)$ is a de Morgan algebra and $(\circ_g)$ g ε G is a family of unary monotone operations on M satisfying the following:
1) $\circ_{g \vee h}\, x = \circ_g\, x \vee \circ_h x$
2) $\circ_{g \wedge h}\, x = \circ_g\, x \wedge \circ_h x$
3) $\circ_{g \circ h}\, x = \circ_g (\circ_h x)$
4) $\circ(g^d)\, x = (\circ_g\, x^d)^d$

To obtain a module over a game algebra,
use $\circ_g x = g \circ x$
Definition: A game module is *separable* if for all distinct elements g and h of G there is an $x \in M$ such that $\circ_g x \neq \circ_h x$
The theorem is proved via the following three steps:

[10] Yde Venema, *Representation of Game Algebras,* Studia Logica 75:244 , 2003

1) Every game algebra seen as a module over itself can be embedded in a separable module over itself.
2) If M is a separable module over G then G is isomorphic to some monotone outcome algebra over M
3) Any monotone outcome algebra can be embedded in a board algebra

Notation: Given a distributive lattice D, then $a\,\hat{} = \{p| a \in p\}$ where p must be a prime filter

There will be an association between every operation of the form $\circ_g x$ and a monotone outcome relation Q_g (to simplify notation, we write Q_g as opposed to $Q_{\circ g}$)on the set of prime filters of M

Main Technical Lemma: Let $(M, \vee, \wedge, d, \circ_g)$ $g \in G$ be a module over game algebra G and let g and h be elements of G, then

We have

$Q_{g\vee h} = Q_g \cup Q_h$

$Q_{g\wedge h} = Q_g \cap Q_h$

$Q_{g\circ h} = Q_g \circ Q_h$

if $\circ_g x \neq \circ_h x$ for some x then $Q_g \neq Q_h$

Proof: The proof makes use of prime filters so first some background:

Let D = $(D, \vee, \wedge)$ be a distributive lattice. A filter is a subset F of D which is upward closed and closed under meets

A filter is prime when $a \vee b \in F \Rightarrow a \in F$ or $b \in F$

Let B_D denote the set of prime filters of D

Also, given any set of prime filters T let F_T denote the set of elements a of D such that for all p in T, $a \in p$

Venema will define an important outcome relation on the board of prime filters of D by in the following way: Given a monotone lattice expansion $(D, \vee, \wedge, \circ)$ (where $\circ$ is a monotone relation on the distributive lattice) let $Q_\circ$ be the outcome relation on the board of prime filters of D given by $p\, Q_\circ\, T$ iff $\circ a \in p$ for all a in F_T (Note that $\circ a$ is not $\circ_a$ which denotes a function where the former denotes the value of a function)

Observation: $p\, Q_\circ\, a\hat{}$ iff $\circ a \in p$ (the first direction is trivial, for the backwards direction, suppose that $\circ a \in p$ and let $b \in F_{a\hat{}}$ so we know $a\hat{} \subseteq b\hat{}$ and so $a \leq b$ and by monotonicity and the proper of filters being upwards closed, the result follows)

Lemma:

$p\, Q_\circ \cap_{a \in A} a\hat{}$ iff $\circ a \in p$ for all $a \in A$

Now to see how the proof of the technical lemma will go:

Let p be an arbitrary prime filter and T an arbitrary set of prime filters

case 1) T = a^ for some a

case 2) T is an arbitrary set

for case 1): $pQ_{g\vee h}\, T$ iff $\circ_{g\vee h} a \in p$ iff $\circ_g a \vee \circ_h a \in p$ iff (since p is prime) $\circ_g a \in p$ or $\circ_h a \in p$ iff $pQ_g\, a\hat{}$ or $pQ_h\, a\hat{}$ iff $p(Q_g \cup Q_h)\, a\hat{}$

case 2): $pQ_{g\vee h}\, T$ then $\circ_{g\vee h} a \in p$ for all a in F_T, so by the proof of the first case for all $a \in F_T$, $\circ_g a \in p$ or $\circ_h a \in p$. We claim $p\, Q_g\, T$ or $p\, Q_h\, T$, for deny toward a contradiction, then there exist $b_g, b_h, \in F_T$ such that $\circ_g b_g \notin p$ and $\circ_h b_h \notin p$; now define $b = b_g \wedge b_h$, and since $b \leq b_g$ we have $\circ_g b \notin p$ by monotonicity of $\circ_g$ and upward closure of prime filters but then $\circ_{g\vee h} b \notin p$, a contradiction; for the backward direction, assume $p(Q_g \cup Q_h)\, T$ so $pQ_g\, T$ or $pQ_h T$ WLOG, $pQ_g\, T$, then given any $a \in F_T$, $\circ_g a \in p$ and hence $\circ_{g\vee h} a \in p$ since $\circ_{g\vee h} a \geq \circ_g a$

The other cases in the technical theorem are similar

Van Benthem also discusses imperfect information games. Here, players do not have perfect recall as to what as to what moves have been played. Knowledge operators are added to a simple modal language to reason about the game. Van Benthem points out that a player V may know that de dicto she has a winning strategy but not de re. Note that in general a modal sentence φ is de re if there exists some formula γ that occurs in φ which consists of a modal operator followed a formula containing either a variable free in γ or an individual constant. For example, $\Box\forall x(Fx \vee Gx)$ is de dicto while $\exists x \Diamond Gx$ is de re.

In an example of an imperfect information game, F has a choice to play c or d and V has a choice to play a or b. If F plays c then V's play of a results in win and a play of b results in a loss but if F plays d, V's play of a results in a loss and V's play of b results in a win.

K_V(<a> win $\vee$ <b>win)

(this notation comes from Parikh[11]) " K knows a wins or b wins "so that V knows de dicto that she has a winning move however not de re for $\neg K_V$<a> win $\wedge$ $\neg K_V$<b> "K does not know that a is winning and K does not know that b is winning"

However, it seems that it is also possible to consider games where the moves are known but the strategies or the thinking of the opponent is not.

Van Benthem also discusses bisumulations between game models. A game model $(S,\{E_g|\ g \in \Gamma\},V)$ corresponds to a Kripke model $K = (S, \{R_g|\ g \in \Gamma\},V)$ where we define $s\, E_g\, X$ iff there exists $t \in X$ such that $s\, E_g\, t$ and there is a standard sematic definition of <g>φ "player 1 has a φ-strategy in g by K,s|= <g>φ iff there is some t in S such that $s\, R_g\, t$ and K,t|= φ[12]

[11] *Rohit Parikh, The Logic of Games and its Applications, Annals of Discrete Mathematics(1985) 111-140*

[12] *Marc Pauly and Rohit Parikh, Game Logic- An Overview, Studia Logica 75:165-182*

Definition: (Bisumulation): Given game models M = (S,{E_g| g ε Γ},V) and M' = (S',{E'_g| g ε Γ},V) then $\leftrightarrow$ $\subseteq$ S x S' is a bisumulation between M and M' iff for any s $\leftrightarrow$ s' we have that

1) s $\in$ V(p) iff s' $\in$ V'(p) for all p $\in$ Φ
2) for all g $\in$ Γ and X $\subseteq$ S: If s E_g X then $\exists$ X' $\subseteq$ S' such that s' E_g X' and $\forall$x' $\in$ X' $\exists$ x $\in$ X : x $\leftrightarrow$ x'
3) for all g $\in$ Γ and X' $\subseteq$ S': If s' E'_g X' then $\exists$ X $\subseteq$ S such that s E_g X and $\forall$x $\in$ X $\exists$ x' $\in$ X' : x $\leftrightarrow$ x'[13]

Response to Van Benthem's Second Question:

1 Necessitation for a way to classify strategies
2 Higher Order Modal Logic to make sense of the classification
3 Necessitation for a way to classify the thinking of players

Note: Games are more complicated than the Van Benthem axiomatization. Players are intelligent agents who have knowledge that they are in a game and realize that there are strategies for the game, and they are aware that the opponent may be playing according to some strategy. Hence, they may make moves to induce the other player to employ a strategy against a perceived strategy (this means that the first player is playing a meta strategy. In this sense, strategies are no longer simply only functions on $X^w \rightarrow X$).
It seems apparent to me that the dynamic aspect of a *game* Van Benthem alludes relates to the interaction between intelligent agents who are aware of the game being played and various possible strategies for playing the game, agents who have a metalevel view of the game and can plan strategies based on their perceived understanding of the reasoning of their opponent.
For example, in a famous example of military strategy, Player I (the Allies) places military equipment close to Calais so that Player II (the Germans) will think an attack is coming there and not at Normandy. How does player II (the Germans) respond? Player II may react in just such a manner. Or perhaps player II reasons that player I wants them to think an attack is coming to Calais and so the real attack is coming to Normandy, or perhaps player II reasons that player I wants them to think that the real attack is coming to Normandy because they believe that player I has anticipated that player II will see player one's play of the game as a ruse to invoke a response from player II to protect Normandy when indeed the actual attack is coming to Calais. However, maybe player I reasoned that Player II would think exactly this way, and so player I is indeed planning to attack Normandy, and such nesting of strategic reasoning can go on ad infinitum. Poker is another game where a strategy may be employed by one player which includes consideration of the opponents ability to interpret what strategy is actually being indicated by a particular move.

[13] *Marc Pauly and Rohit Parikh, Game Logic- An Overview, Studia Logica 75:165-182*

Classification of intent of players in a game to which there is no clear winning strategy other than to anticipate the opponents move as in the first example:
Meta-Level 0) at the er level, player II reacts to the moves of player I and playes move m according to a strategy which does not anticipate the thinking of player I. This is a 0-level reason for making move m.
Meta-Level 1) A player reacts to the moves of the other player by considering a response at level 0 as part of the reasoning process of the other player and makes a move m to thwart the other player
Meta-Level 2) A player reacts to the moves of the other player by considering a response at level 1 as part of the reasoning process of the other player

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Meta-Level i + 1) In general, a player reacts to the moves of the other player by considering a response at level i as part of the reasoning process of the other player
Here we might refer to a strategy with intent of influencing or interpreting an opponents thinking as < τ , σ, p, Θ > where τ and σ are functions $X^{<w} \rightarrow X$ and $\sigma = \tau|$ n for some n and p is a current position and Θ is a k-metalevel reason for provoking a move m at stage n + 1 of the game, where σ is winning strategy for p ^m.
Or imagine a game of chess in which a winning strategy is either not clear to either player or where the game can only be drawn with perfect play but player one knows that player II has a predilection to interpret a particular move as a perceived strategy in such a way that there is a alternate strategy following the expected response from II that is indeed winning. Hence, it is apparent there needs to be a classification of strategies. Which strategies are devised on the basis of a feigned strategy in position s? And which sets of strategies are devised on the basis of a feigned strategy from some position, necessitating properties of properties of strategy. We need to also know from which positions are such properties of properties of strategies possible, so a third order modal logic or even higher order modal logic is needed.

Classification of Strategies:
How can we classify strategies using higher order modal logics?

1) Strategies can be classified at the er level as winning or losing for a certain player, and a pair of strategies could for example be classified at the er level as being equivalent at some finite stage of play
2) A strategy may have the property that it will provide player 1 with a move such that there exists a strategy which player II will adopt with high probability following such a move by player 1 such that there exists a strategy winning for I and such that this strategy is equivalent at some finite stage to the adopted strategy by I. This is easily expressible in a second order modal logic.
3) Properties of strategies can be classified as belonging to the same property of properties if the manner of classification in 2) is similar for example, and this is expressible in a third order modal logic.

Example: Given σ and τ as strategies, $\Diamond \exists R \exists P [R(P) \wedge P(\sigma,\tau)]$ is the third order modal logic statement that expresses the notion that from the current game position it is possible to move in such a way that there is a property of relations R and binary relation of strategies P such that P has property R and σ and τ belong to P.

In *Godel's Ontological Argument*[14] I developed the formal syntax of Third Order Modal Logic with a property abstraction operator and proved a
Completeness Theorem of third order modal logic for faithful models: A completeness theorem for third order modal logic useful in the setting of games and strategies could be adaptable from this more general completeness theorem for third order modal logic.

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