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q-SIMPSON'S RULES OF QUANTUM CALCULUS

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Abstract- This paper is devoted to the derivation of *q*-Simpson's rules for numerical integration. Instead of the classical Newton divided differences used to establish the classical Simpson's method, we apply the Jackson *q*-differences in our new approach. A rigorous error analysis is given without assuming differentiability conditions on the integrands. Illustrative examples with comparison with classical results are also demonstrated.

Keywords- *q*-calculus, *q*-integration, Simpson's rule, *q*-Simpson's rule, *q*-Taylor series.

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1. Introduction

The *q*-calculus, or quantum calculus, cf. e.g. [9,13,20] has received more attenstion in the last two decades, especially, analogues of classical results, its application in physics as well as approximation theory, cf. e.g. [10,21,22]. So, it is a desirable task to derive *q*-analogues of classical results in a self dependent manner, within the frame of *q*-calculus. This paper aims to develop a new method for computing integrals numerically. This technique is *q*-analogues of the well known Simpson's rule. In this setting *q*-type Taylor theroems are employed. The *q*-Taylor series has been introduced first by Jackson [17] as the following

$$f(x)=\sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}D_q^n f(a)[x-a]_n, \tag{1.1}$$

where q is a positive number with $0<q<1$, D_q is the *q*-difference operator defined by Jackson [18] and the *q*-shifted factorial is defined by

$$[x-a]_n=\begin{cases}1, & n=0,\\ (x-a)\ (x-qa)\ \cdots\ (x-q^{n-1}a), & n\geq 1.\end{cases}$$

Rigorous definitions are given below. Al-Salam and Verma [3], introduced another *q*-Taylor series

$$f(x)=\sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2}\frac{D_q^n f(aq^{-n})(1-q)^n}{(q;q)_n}[x-a]_n. \tag{1.2}$$

Since D_q^n can be expressed in terms of $f(x), f(qx), \dots, f(xq^n)$, then both series can be considred as interpolation ones. However, neither papers contain any proofs of these expansions. Both series are derived formally by assuming the expandability of the functions and then computing the corresponding coefficients. Annaby and Mansour have given analytic proofs of both expansions [4].

In this paper we will use the *q*-type Taylor theroems to derive a *q*-analog of Simpson's rule to compute integrals numerically, cf. [4, 7-8]. Therefore we will use finite *q*-Taylor polynomials with *q* -integral remainders. Since we have two of such polynomials we will derive two different rules. Each rule is a family of uncountable rules since $q\in(0,1)$. Composite rules are also given. In the next section we will define the necessary notations and state the important results for our investigations. Section 3 contains the main results of this paper. We derive two *q*-type Simpson's rules to compute integrals numerically. Rigorous error estimates are proved without assuming any differentiability conditions on the integrands. This is another advantage of using *q*-differences. The last section is fully devoted to numerical examples with comparisons.

2. *q*-Notations and Results

This section involves notations and results that will be needed in the sequel. By a *q*-geometric set $A\subseteq\mathbb{R}$, it is meant that $qx\in A$ for all $x\in A$. Intervals containing zero are example of *q*-geometric sets.

The *q*-difference operator [17], is defined by the following.

Definition 2.1: Let $f(x)$ be a function defined on a *q*-geometric set A. The *q*-difference operator is defined for $x \in A, x \neq 0$ to be

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{2.1}$$

Since $x \neq qx$, then $D_q f(x)$ always exists for $x \in A, x \neq 0$ and it is called the *q*-derevative of *f* at *x*. The *q*-derevative at zero is defined to be

$$D_q f(0) := \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A, \tag{2.2}$$

provided that the limit exists without depending on *x*. The *q*-derivative at zero is defined in many literature to be the $f'(0)$ if it exists [11,13].

Definition 2.2: A function *f* defined on a *q*-geometric set $A, 0 \in A,$ is said to be *q*-regular at zero if $\lim_{n\to\infty} f(xq^n) = f(0)$ for every $x \in A.$

In some occasions *q*-regularity at zero plays the role of continuity in the classical sense. Notice that continuity at zero implies *q*-regularity at zero, but the converse is not necessarily true. For example, cf. [1], the function $f:[0,1] \to \square$,

$$f(x) = \begin{cases} 1, & x = a_n = \dfrac{1}{\sqrt{n}}, \quad \text{n is prime,} \\ x, & \text{otherwise.} \end{cases} \tag{2.3}$$

Is *q*-regular at zero for any rational *q,* but it is not continuous at zero. The relationship between the classical derivative and the *q* - derivative can be explained as follows. If *A* contains a neighborhood of a point *x,x≠0* and *f* is differentiable at *x* then $\lim_{q\to 1^-} D_q f(x) = f'(x)$. Also if $x = 0$ and $f'(0)$ exists, then also $D_q f(0) = f'(0)$. Moreover if $D_q f(0)$ exists, then *f* is *q*-regular at zero. Nevertheless, (2.3) indicates that the existence of $D_q f(0)$ does not imply continuity at zero. The *q*-integration on intervals of the form *[x,*1)*, x*>0 had been introduced by Jackson and Hahn [5].

Definition 2.3: The *q*-integration over $[0,\infty)$ using the division points $\{q^n, n \in \square\}$

is defined by

$$\int_0^\infty f(x) d_q x := (1-q) \sum_{-\infty}^{\infty} q^n f(q^n). \tag{2.4}$$

Definition 2.4: The *q*-integration over [*x*,1), *x*>0 using the division points $\{xq^{-n}\}_{n=1}^\infty$

is defined by the formula

$$\int_x^\infty f(t) d_q t := x(1-q) \sum_{n=1}^\infty q^{-n} f(xq^{-n}). \tag{2.5}$$

Jackson in [16] introduced an integral denoted by $\int_a^b f(x) d_q x.$

Definition 2.5: The *q*-integration for $f(x)$ on a *q*-geometric set A to be, $a, b \in A,$

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \tag{2.6}$$

where

$$\int_0^x f(t) d_q t := (1-q) \sum_{n=0}^\infty xq^n f(xq^n), \quad x \in A, \tag{2.7}$$

provided that the series at the right-hand side of (2.7) converges at $x = a \text{ and } b$. Although one can prove some algebraic properties of *q*-integration in a straight forward manner, some properties do not hold. For instance the inquality

$$\left| \int_a^b f(t) d_q t \right| \leq \int_a^b |f(t)| d_q t, \quad 0 \leq a \leq b < \infty, \tag{2.8}$$

is not always true. Obviously such an inquality would play an important role in deriving error estimates of numerical methods if it exists. To see this, define the function $g:[0,1] \to \square$ to be

$$g(x) = \begin{cases} \dfrac{1}{1-q}(4q^{-n}x - (1+3q)), & q^{n+1} \leq x \leq \dfrac{q^n(1+q)}{2}, n \in \square, \\ \dfrac{4}{1-q}(-xq^{-n} + 1) - 1, & \dfrac{q^n(1+q)}{2} \leq x \leq q^n, n \in \square, \\ 0, & x = 0. \end{cases} \tag{2.9}$$

Clearly g is *q*-integrable [0,1] and

$$g(q^n) = -1 \text{ and } g\left(\frac{1+q}{2} q^n\right) = 1, \quad n \in \square.$$

Direct calculations yield

$$\int_{\frac{1+q}{2}}^1 h(t) d_q t = -\frac{3+q}{2}, \quad \int_{\frac{1+q}{2}}^1 |h(t)| d_q t = \frac{1-q}{2}.$$

Consequently,

$$\left| \int_{\frac{1+q}{2}}^1 h(t) d_q t \right| > \int_{\frac{1+q}{2}}^1 |h(t)| d_q t.$$

However, inequality (2.8) holds when $a=0$ or when $a,b \in I, a<b,$ the forms have $a = xq^n \quad b = xq^m;\ n,m \in \mathbb{N}$.

The q-shifted factorial for $n \in \mathbb{N} = \{0,1,...\},\ a \in \mathbb{C}$ is defined by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k). \tag{2.10}$$

The limit, $\lim_{n\to\infty}(a;q)_n$, exists since $0<q<1.$ It will be denoted by $(a;q)_\infty.$ The q-Gamma function [11, 19] is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z \in \mathbb{C},\ |q|<1 \tag{2.11}$$

where we take the principal branches for q^z and $(1-q)^{1-z}$. One can easily deduce

$$\Gamma_q(n+1) := \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

For our purpose we consider functions defined on $[0,\alpha]$. Let $\varphi_n(x,a),\ x,a \in \mathbb{R}$, be the polynomial

$$\varphi_0(x,a) := 1, \quad \varphi_n(x,a) := \begin{cases} x^n(a/x;q)_n, & x \neq 0, \\ (-1)^n q^{\frac{n(n-1)}{2}} a^n, & x = 0. \end{cases} \tag{2.12}$$

Let f be defined on $[0,\alpha]$ and $n \in \mathbb{N}^+,$ such that the q-derivative of f up to order n exist at zero and $D_q^n f(x)$ is q-integrable on $[0,\alpha]$. Then for a fixed $a \in [0,\alpha],\ x \in [0,\alpha]$ we have the following two q-Taylor formulas

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)}\varphi_k(x,a) + \frac{1}{\Gamma_q(n)}\int_a^x \varphi_{n-1}(a,qt) D_q^n f(t)\,d_q t. \tag{2.13}$$

$$f(x) = \sum_{k=0}^{n-1} (-1)^k q^{-k(k-1)/2} \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)}\varphi_k(a,x) + \frac{1}{\Gamma_q(n)}\int_{aq^{-n+1}}^x \varphi_{n-1}(a,qt) D_q^n f(t)\,d_q t, \tag{2.14}$$

see [4, 8] for proofs and references. In [14, 15] Ismail and Stanton derived with analytic proofs q-type Taylor series for entire functions of q-exponential growth, [26]. Ismail and Stanton's results stand for the Askey-Wilson difference operator.

Lo'pez, et al. [23] using [24]established sufficient conditions for the convergence of Ismail and Stanton's q-Taylor series, but not necessarily to the original function. In [4], analytic proofs of for Jackson and Al-Salam-Verma q-Taylor series are given using q-Cauchy integral formulas, see e.g. [2,8].

q- Simpson's Rules

In the following we are considering the interval $[a,b],\ a>0.$ The classical Simpson's rule states that for $f \in C^{(4)}[a,b]$, the class of functions which are continously differentiable up to order four, we have

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^5}{90}f^{(4)}(\xi), \tag{3.1}$$

where $x_0 = a, x_1 = a+h, x_2 = b, h = \frac{b-a}{2}$ and ξ lies in (a,b). Based on this, the composite Simpson's rule is derived when splitting the interval [*a*,*b*] into *n* even number of subintervals with equal lengths via an equidistant partition

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b \tag{3.2}$$

Then, the composite Simpson's rule is given by

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j}-1) + f(x_n)\right] - \frac{h^4}{180}(b-a)f^4(\xi), \tag{3.3}$$

where $x_j = a + jh$ for $j = 0,1,...,n-1,n$ with $h=(b-a)/n$ [5,7,12]. In the following we will derive q-Simpson's rules which depend on the q-differences instead of the clasical ones. With the same numbers of nodes, the new techique gives better results. The proposed rules are families of rules with the same stepsize. The present work, which is the first in this direction, as far as we know, indicates that the new technique enriches numerical integration techniques. Moreover in the new setting, no differentiability conditions are imposed on the function, which may badly behave. In what follows we consider functions defined on [0,*b*] and we will derived two q-Simpson's rules to compute integrals on [*a*,*b*], 0<a<b. The first rule is a backward rule and the second is a forward one. Before deriving our rules, we state the following q-mean value theorem, taken from [25]. It will be needed in establishing error estimates. It states that if $f(x)$ and $g(x)$ are continous on $[0,b]$ then there exists $\hat{q} \in (0,1)$ such that

$$\forall q \in (\hat{q},1), \exists \xi \in (a,b): \int_a^b f(x)g(x)d_q x = g(\xi)\int_a^b f(x)d_q x.$$

Now start with deriving a backward q-Simpson's rule. Letting n=3 in (2.13) and simplifying φ_k we obtain

$$f(x) = f(b) + D_q f(b)(x-b) + D_q^2 f(b)\frac{(x-b)(x-qb)}{1+q} + \frac{1}{1+q}\int_b^x (x-qt)(x-q^2t)D_q^3 f(t)d_q t. \tag{3.4}$$

Theorem 3.1: Let $f:[0,b] \to \mathbb{R}$ be such that $D_q^3(0)$ exists. Then

$$\int_a^b f(x)dx = BQSimR(f,h,q) + E_{BQ}(f,h,q) \tag{3.5}$$

where $h=(b-a)/2,$

$$BQSimR(f,h,q)=f(b)\left[h-\frac{h^2}{2b(1-q)}-\frac{h^2q(a-qb)}{2b^2(1-q)^2(1+q)}-\frac{h^3}{6b^2(1-q)^2(1+q)}\right]$$
$$+f(qb)\left[\frac{h^2}{2b(1-q)}+\frac{h^2(a-qb)}{2qb^2(1-q)^2}+\frac{h^3}{6qb^2(1-q)^2}\right]$$
$$+f(q^2b)\left[\frac{-h^2(a-qb)}{2qb^2(1-q)^2(1+q)}-\frac{h^3}{6qb^2(1-q)^2(1+q)}\right] \tag{3.6}$$

and

$$E_{\mathrm{BQ}}(f,h,q)=\frac{1}{1+q}\int_a^b\int_b^x(x-qt)(x-q^2t)D_q^3f(t)d_qt\,dx. \tag{3.7}$$

Moreover, if f is continuous; $D_q^3f(x)$ is bounded on [0,*b*] and *q*→1, then the error has the estimate

$$E_{\mathrm{BQ}}(f,h,q)=\frac{M}{(1+q)(1+q+q^2)}h^4$$

where $\left|D_q^3f(x)\right|\le M$ on [0,*b*] (3.8)

Proof: By integrating (3.4), we have

$$\int_a^b f(x)dx=\int_a^b f(b)+D_qf(b)(x-b)+D_q^2f(b)\frac{(x-b)(x-qb)}{1+q}\,dx$$
$$+\frac{1}{1+q}\int_a^b\int_b^x(x-qt)(x-q^2t)D_q^3f(t)d_qt\,dx. \tag{3.9}$$

Computing and simplifying the first integral of the previous equation directly leads to the *q*-Simpson's rule

$$\int_a^b f(b)+D_qf(b)(x-b)+D_q^2f(b)\frac{(x-b)(x-qb)}{1-q}dx=$$
$$f(b)\left[h-\frac{h^2}{2b(1-q)}-\frac{h^2q(a-qb)}{2b^2(1-q)^2(1+q)}-\frac{h^3}{6b^2(1-q)^2(1+q)}\right]$$
$$+f(qb)\left[\frac{h^2}{2b(1-q)}+\frac{h^2(a-qb)}{2qb^2(1-q)^2}+\frac{h^3}{6qb^2(1-q)^2}\right]$$
$$+f(q^2b)\left[\frac{-h^2(a-qb)}{2qb^2(1-q)^2(1+q)}-\frac{h^3}{6qb^2(1-q)^2(1+q)}\right]. \tag{3.10}$$

Now we estimate the error. Since *q*→1, we can apply the *q*-mean value theorem stated above, and consequently

$$\int_b^x(x-qt)(x-q^2t)D_q^3f(t)d_qt=D_q^3f(\xi(x))\int_b^x(x-qt)(x-q^2t)d_qt \tag{3.11}$$

for some $\xi(x)\in(x,b)$. From the definition of *q*-integral and direct calculations we obtain

$$\int_b^x(x-qt)(x-q^2t)\,d_qt=\int_0^x(x-qt)(x-q^2t)d_qt-\int_0^b(x-qt)(x-q^2t)d_qt$$
$$=x(1-q)\sum_{k=0}^{\infty}q^k(x-xq^{k+1})(x-xq^{k+2})-b(1-q)\sum_{k=0}^{\infty}q^k(x-bq^{k+1})(x-bq^{k+2})$$
$$=\frac{1}{1+q+q^2}(x-b)(x-bq)(x-bq^2). \tag{3.12}$$

Combining (3.11), (3.12) and (3.7) yields,

$$E_{\mathrm{BQ}}(f,h,q)\le\frac{1}{(1+q)(1+q+q^2)}\int_a^b\left|D_q^3f(\xi(x))\right|\left|(x-b)(x-qb)(x-q^2b)\right|dx$$
$$\le\frac{M}{(1+q)(1+q+q^2)}\int_a^b h^3dx=\frac{M}{(1+q)(1+q+q^2)}h^4, \tag{3.13}$$

where *q* is closer to one such that a≤qb<b

We call the previous rule backward because it is derived in terms of $f(b),f(qb)$ and $f(q^2b)$. It should be noted that the condition *q*→1 is not very much restrictive because *h* is supposed to be small to guarantee that $qb,q^2b\in[a,b]$. Therefore

$b>qb>a\Rightarrow1>q>\dfrac{a}{b}=\dfrac{a}{a+h}\to1$ as $h\to0$.

The composite rule directly follows when we have the partition (3.2) to be.

Corollary 3.2: Let $f:[0,b]\to\mathbb{R}$ be such that $D_q^3(0)$ exists. Then

$$\int_a^b f(x)dx=CBQSimR(f,h,q)+E_{\mathrm{CBQ}}(f,h,q) \tag{3.14}$$

where $h=(b-a)/n$

$$CBQSimR(f,h,q)=\sum_{k=1}^{n}f(x_k)\left[h-\frac{h^2}{2x_k(1-q)}-\frac{h^2(a-qx_k)}{2x_k^2(1+q)(1-q)^2}-\frac{h^3}{6x_k^2(1+q)(1-q)^2}\right]$$
$$+f(qx_k)\left[\frac{h^2}{2x_k(1-q)}+\frac{h^2(a-qx_k)}{2qx_k^2(1-q)^2}+\frac{h^3}{6qx_k^2(1-q)^2}\right]$$
$$+f(q^2x_k)\left[\frac{-h^2(a-qx_k)}{2qx_k^2(1+q)(1-q)^2}-\frac{h^3}{6qx_k^2(1+q)(1-q)^2}\right] \tag{3.15}$$

and

$$E_{\mathrm{CBQ}}(f,h,q)=\frac{1}{1+q}\sum_{k=1}^{n}\int_a^b\int_{x_k}^x(x-qt)(x-q^2t)D_q^3f(t)d_qt\,dx \tag{3.16}$$

Moreover, if f is continuous; $D_q^3f(x)$ is bounded on [0,*b*] and *q*→1, then the error has the estimate

$$E_{\mathrm{CBQ}}(f,h,q)\le\frac{M(b-a)}{(1+q)(1+q+q^2)}h^3$$

where $\left|D_q^3f(x)\right|\le M$ on $[0,b].$ (3.17)

Next we derive a forward rule in terms of $f(a)$, $f(q^{-1}a)$ and $f(q^{-2}a)$. Therefore *q* should be chosen closer to one again to make sure that $q^{-1}a,q^{-2}a\in[a,b].$ Indeed

$a<q^{-1}a<b\Rightarrow1>q>\dfrac{a}{b}=\dfrac{a}{a+h}\to1$ as $h\to0$.

Let $n=3$ in (2.14). Then

$$f(x)=f(a)-D_q f(q^{-1}a)(x-a)+q^{-1}D_q^2 f(q^{-1}a)\frac{(x-a)(x-qa)}{1+q}$$
$$+\frac{1}{1+q}\int_{q^{-1}a}^{x}(x-qt)(x-q^2t)D_q^3 f(t)d_qt. \tag{3.18}$$

Theorem 3.3: Let $f:[0,b]\to\mathbb{R}$ be such $D_q^3(0)$ exists.Then

$$\int_a^b f(x)dx = FQSimR(f,h,q)+E_{FQ}(f,h,q), \tag{3.19}$$

where $h=(b-a)/2$

$$FQSimR(f,h,q)=f(a)\left[h-\frac{h^2}{2a(q^{-1}-1)}+\frac{(b-aq)h^2}{2a^2(1+q)(q^{-1}-1)^2}-\frac{h^3}{6a^2(1+q)(q^{-1}-1)^2}\right]$$
$$+f(q^{-1}a)\left[\frac{h^2}{2a(q^{-1}-1)}-\frac{(b-aq)h^2}{2a^2(q^{-1}-1)^2}-\frac{h^3}{6a^2(q^{-1}-1)^2}\right]$$
$$+f(q^{-2}a)\left[\frac{(b-aq)h^2}{2a^2q^{-1}(1+q)(q^{-1}-1)^2}-\frac{h^3}{6a^2q^{-1}(1+q)(q^{-1}-1)^2}\right], \tag{3.20}$$

and

$$E_{FQ}(f,h,q)=\frac{1}{1+q}\int_a^b\int_{aq^{-2}}^{x}(x-qt)(x-q^2t)D_q^3 f(t)d_qt\,dx. \tag{3.21}$$

Moreover, if f is continuous; $D_q^3 f(x)$ is bounded on [0,*b*] as *q*→1 then the error has the estimate

$$E_{\text{FQ}}(f,h,q)=\frac{M}{q^3(1+q)(1+q+q^2)}h^4$$

where $\left|D_q^3 f(x)\right|\le M$ on [0,*b*].

Proof: From (3.18), we have

$$\int_a^b f(x)dx=\int_a^b f(a)-D_q f(q^{-1}a)(a-x)+D_q^2 f(q^{-2}a)\frac{q^{-1}(x-a)(x-qa)}{1+q}dx$$
$$+\frac{1}{1+q}\int_a^b\int_{q^{-2}a}^{x}(x-qt)(x-q^2t)D_q^3 f(t)d_qt\,dx. \tag{3.23}$$

Simple manipulations imply *q*-Simpson's rule

$$\int_a^b f(a)+D_q f(q^{-1}a)(x-a)+D_q^2 f(q^{-2}a)\frac{q^{-1}(x-a)(x-qa)}{1+q}dx=$$
$$f(a)\left[h+\frac{h^2}{2a(q^{-1}-1)}+\frac{(b-aq)h^2}{2a^2(1+q)(q^{-1}-1)^2}-\frac{h^3}{6a^2(1+q)(q^{-1}-1)^2}\right]$$
$$+f(q^{-1}a)\left[\frac{-h^2}{2a(q^{-1}-1)}-\frac{(b-aq)h^2}{2a^2(q^{-1}-1)^2}+\frac{h^3}{6a^2(q^{-1}-1)^2}\right]$$
$$+f(q^{-2}a)\left[\frac{(b-aq)h^2}{2a^2q^{-1}(1+q)(q^{-1}-1)^2}-\frac{h^3}{6a^2q^{-1}(1+q)(q^{-1}-1)^2}\right]. \tag{3.24}$$

We estimate the error as in the previous theorem. Suppose that *q*→1 and use the mean value theorem of [25], then there exists $\xi(x)\in(x,b)$ such that

$$\int_{aq^{-2}}^{x}(x-qt)(x-q^2t)D_q^3 f(t)d_qt = D_q^3 f(\xi(x))\int_{aq^{-2}}^{x}(x-qt)(x-q^2t)d_qt. \tag{3.25}$$

From the definition of *q*-integrals we obtain

$$\int_{aq^{-2}}^{x}(x-qt)(x-q^2t)d_qt=\int_0^x(x-qt)(x-q^2t)d_qt-\int_0^{aq^{-2}}(x-qt)(x-q^2t)d_qt$$
$$=x(1-q)\sum_{k=0}^{\infty}q^k(x-xq^{k+1})(x-xq^{k+2})-aq^{-2}(1-q)\sum_{k=0}^{\infty}q^k(x-aq^{k-1})(x-aq^k)$$
$$=\frac{1-q}{q^3(1-q^3)}(a-x)(a-qx)(a-xq^2). \tag{3.26}$$

Substitution from (3.26) and (3.25) in (3.21) yields,

$$E_{\text{FQ}}(f,h,q)\le\frac{1}{q^3(1+q)(1+q+q^2)}\int_a^b\left|D_q^3 f(\xi(x))\right|\left|(a-x)(a-xq)(a-xq^2)\right|dx$$
$$\le\frac{M}{q^3(1+q)(1+q+q^2)}h^4. \tag{3.27}$$

As indicated before the condition *q*→1 is not restrictive here too. Also the composite rule for the forward rule with respect to the partition (3.2) will be the following.

Corollary 3.4: Let $f:[0,b]\to\mathbb{R}$ be such that $D_q^3(0)$ exists. Then

$$\int_a^b f(x)dx=CFQSimR(f,h,q)+E_{\text{CFQ}}(f,h,q). \tag{3.28}$$

where $h=(b-a)/n$,

$$CFQSimR(f,h,q)=$$
$$f(x_{k-1})\left[h+\frac{h^2}{2x_{k-1}(q^{-1}-1)}+\frac{(b-x_{k-1}q)h^2}{2x_{k-1}^2(1+q)(q^{-1}-1)^2}-\frac{h^3}{6x_{k-1}^2(1+q)(q^{-1}-1)^2}\right]$$
$$+f(q^{-1}x_{k-1})\left[\frac{h^2}{2x_{k-1}(q^{-1}-1)}-\frac{(b-x_{k-1}q)h^2}{2x_{k-1}^2(q^{-1}-1)^2}+\frac{h^3}{6x_{k-1}^2(q^{-1}-1)^2}\right]$$
$$+f(q^{-2}x_{k-1})\left[\frac{(b-x_{k-1}q)h^2}{2x_{k-1}^2q^{-1}(1+q)(q^{-1}-1)^2}-\frac{h^3}{6x_{k-1}^2q^{-1}(1+q)(q^{-1}-1)^2}\right] \tag{3.29}$$

and

$$E_{\text{CFQ}}(f,h,q)=\frac{1}{1+q}\sum_{k=1}^{n}\int_a^b\int_{q^{-2}x_{k-1}}^{x}(x-qt)(x-q^2t)D_q^3 f(t)d_qt\,dx. \tag{3.30}$$

Moreover, if f is continuous; $D_q^3 f(x)$ is bounded on [0,*b*] and *q*→1 then the error has the estimate

$$E_{\text{CFQ}}(f,h,q)=\frac{M(b-a)}{q^3(1+q)(1+q+q^2)}h^3$$

where $\left|D_q^3 f(x)\right|\le M$ on [0,*b*] (3.31)

4. Numerical Results

In this section we present the results for some numerical experiments. We apply these methods to two numerical examples. In each case, the approximate solution and the maximum absolute error between the exact solution and the approximate solution were given. The results are presented in [Table-1] and [Table-2].

The numerical methods tested are as follows.

Simp R = Simpson's rule (3.3)

CBQ SimpR = Composite backward *q*-Simpson's rule (3.15)

CFQ SimpR = Composite forward *q*-Simpson's rule (3.29)

For our computations we have adopted a *Mathematica* module.

Example 1: Consider the following integral

$$\int_1^2 x^2 \log(x+1)dx = 2.22263 \quad (4.1)$$

Table 1-

n	SimpR	CBQSimpR	CFQSimpR	E_{SimpR}	E_{CBQ}	E_{CFQ}
2	2.22237	2.22463	2.22263	2.58814×10^{-4}	2.00630×10^{-3}	1.23423×10^{-6}
4	2.22261	2.22284	2.22263	1.68305×10^{-5}	2.07635×10^{-4}	1.90956×10^{-8}
8	2.22263	2.22263	2.22263	1.06348×10^{-6}	3.56321×10^{-7}	5.61018×10^{-11}
16	2.22263	2.22263	2.22263	6.66548×10^{-8}	3.55444×10^{-6}	2.62865×10^{-11}
20	2.22263	2.22263	2.22263	2.73111×10^{-8}	6.13079×10^{-6}	2.62865×10^{-11}

Example 2: Consider the following integral.

$$\int_1^3 e^{3x} \sin 3x dx = -1724.97 \quad (4.2)$$

Table 2-

n	SimpR	CBQSimpR	CFQSimpR	E_{SimpR}	E_{CBQ}	E_{CFQ}
2	-1155.71	-1724.97	-1064.98	5.69258×10^{2}	1.7259×10^{-8}	6.59984×10^{2}
4	-1623.46	-1724.97	-1072.05	1.01503×10^{2}	2.12256×10^{-6}	6.52919×10^{2}
8	-1717.16	-1724.97	-1620.61	7.80951	7.06473×10^{-9}	1.04357×10^{2}
16	-1724.46	-1724.97	-1652.94	5.07811×10^{-1}	1.01977×10^{-9}	7.20261×10
20	-1724.76	-1724.97	-1652.32	2.08899×10^{-1}	4.95328×10^{-7}	7.26445×10

Conclusions

We implement our numerical methods, as described above, to two examples. The methods give comparable results with those obtained by simpson's rule. In the second example the composite Simpson's rule and CFQSimp rule are inappropriate when integrating a function on an interval that contains both regions with large functional variation and regions with small functional variation. But CBQSimp rule was employed successfully for solving such type of integral. The difficulty in this method, which needs future investigations, is that we need to estimate the value of $q \in (0,1)$ for each *n*.

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