

Designing of a novel mathematical model for the analysis of aids transmission caused by co- infection of HIV with TB

Raghaw Shukla* and Vivek Kumar

*Department of Applied Mathematics, Prabhat Engineering College, Kanpur, raghawshukla79@gmail.com
Department of Applied Mathematics, Krishna Girls Engineering College, Kanpur

Abstract- This manuscript describes a combined SIS-SIRS model with migration for the spread of AIDS caused in co- infection involving TB and HIV is proposed. In modeling the problem, it is assumed that an HIV infected individual's interacts with TB infected persons the former to have HIV & TB. It is assumed further that a fraction of this HIV class becomes AIDS patients. It is also considered that a certain fraction of TB class form a reservoir class, a part of which may recover and becomes susceptible again. The model is analysed by using stability theory of differential equation. It is shown that AIDS epidemic spread faster due to co- infection of HIV with TB.

Introduction

From the first model it contains a new class of patients who were suffering from both T.B. & AIDS. So it has different calculations are included. The increasing incidence of various infectious diseases such as gonorrhoea, T.B., HIV, AIDS, etc. is causing major public health problems in the most developed and developing countries [1]. In general the spread of such diseases in human population depends upon various factors such as the numbers of infectives, susceptible, modes of transmission (social, ecological, and geographical conditions) [2]. An account of modeling and study of epidemic diseases can be found in lectures notes by [3] and books by [4]. Many infectious diseases are spread by direct contact between susceptible and infectives, while some others are spread indirectly by carriers bacteria or vectors [5]. Tuberculosis (TB) is an infectious disease caused by a bacteria called Mycobacterium tuberculosis and affects lung and other parts of the body. In developing countries pulmonary tuberculosis (i.e., T.B.) in the lung accounts more than seventy percent of cases in the world. T.B., which had been controlled to a significant extent earlier, has re-emerged in recent years as one of the leading causes of death. Nearly three million people die every years due to T.B. Any one can become infected with T. B. bacteria but people with H.I.V. are at greater risk of getting infected with this disease. T.B. is transmitted mainly by droplet nuclei when the person suffering from active T.B. breathes, sneezes or coughs in the air [6]. AIDS, the acquired immunodeficiency syndrome is a fatal disease caused by a retrovirus known as the human immunodeficiency virus (HIV) which although attacks many blood cells it causes havoc on the T-cell in the blood by destroying and decreasing their number leading to decline in body's immunity to fight infection. The term AIDS refers to only the last stage of the HIV infection after which death occurs. The AIDS epidemic is sweeping the world both developed and developing nations as HIV

has infected millions of people all over the world without any restriction of nationality, color, caste or religion, etc. HIV infection in adults (75-80%) has been transmitted to one partner through unprotected sexual intercourse when the other partner is infected with HIV. Mother to child transmission (vertical infection) accounts more than 90% of global infection to infants and children. Apart from these, other possible source of infection are blood transfusion, intravenous drug addiction etc. In general sexually transmitted disease (STD) such as gonorrhoea induce little or no acquired immunity upon recovery but in case of HIV immunity decreases [4]. The transmission dynamics of HIV infection is affected by various factors such as latent period, the number of infected/infectious person in the population, having AIDS, type of sexual activity, type of mixing (homogeneous or heterogeneous, various high risk groups, etc. It is noted that in case of HIV infected persons immunity decreases and they can early infected by other diseases such as TB forming a class HIV co infected with TB. Some efforts have been to understand the transmission of co infection of HIV with TB infections. mathematical models, taking into account fixed values for the transmission coefficients, these models dealt with the dynamic simulations and the equilibrium point analysis. However, if the equilibrium point analyzes could be achieved for all finite the transmission coefficients, then it would provides us different scenarios of the interaction between HIV and TB infections. In this paper a model for the transmission of co infection HIV with TB infections and examine some of the epidemiological implications.

A combined sis-sirs model with constant immigration

In this section a combined SIS-SIRS model with constant immigration of human population is proposed and analyzed. In proposing the model, the following assumptions are made.

(1). All HIV infected individuals are considered susceptible to TB infection and, at the same

manner, all TB infected individuals are considered susceptible to HIV.

(2).Due to loss of immunity HIV infected persons get infected with TB after interaction.

(3).There is a separate class of Pearson infected both with HIV and TB. The HIV class also includes persons having co infection. Based on the above assumptions we propose a mathematical model to assess the spread of co infection HIV with TB infections .It is assumed that the population density N(t) is divided into five classes , the susceptible density X(t),the density of population infected only with TB(Y_T),the

density of population infected with HIV(Y_H)a fraction of which may be also infected with TB ,the density of AIDS class(Y_a) and the density of co infected class. The models is proposed as follows,

$$\begin{aligned} \frac{dX}{dt} &= A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + v_1 Y_T \\ \frac{dY_T}{dt} &= \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T \\ \frac{dY_H}{dt} &= \frac{\beta_2 Y_H X}{N} - \frac{\beta_4 Y_T Y_H}{N} - dY_H - \mu Y_H \\ \frac{dY_{TH}}{dt} &= \frac{\beta_3 Y_T Y_H}{N} + \frac{\beta_4 Y_T Y_H}{N} - dY_{TH} - \mu_1 Y_{TH} \\ \frac{dY_a}{dt} &= \mu Y_H + \mu_1 Y_{TH} - dY_a - \alpha Y_a \end{aligned}$$

$$N = X + Y_T + Y_H + Y_{TH} + Y_a$$

Where
so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

in this model , β_1 and β_2 are the contact-rates by which susceptible decrease following contact

with persons having of TB and AIDS and β_3, β_4 be the co-infection constant . The clinical AIDS

developed at the rate μ per year. v_1 is the recovery rate at which TB infectives becomes susceptible again ,d is the natural death rate , α

be death rate due to AIDS. μ_1 be the rate at which co infected class becomes AIDS susceptible.

3. EQUILIBRIUM ANALYSIS

To analyze the model, we consider the following reduced system. Since

$$\begin{aligned} N &= X + Y_T + Y_H + Y_{TH} + Y_a \\ \frac{dY_T}{dt} &= \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_{TH} - Y_a)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1) Y_T \\ \frac{dY_H}{dt} &= \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_{TH} - Y_a)}{N} - \frac{\beta_4 Y_T Y_H}{N} - (d + \mu) Y_H \\ \frac{dY_{TH}}{dt} &= \frac{\beta_3 Y_T Y_H}{N} + \frac{\beta_4 Y_T Y_H}{N} - d Y_{TH} - \mu_1 Y_{TH} \\ \frac{dY_a}{dt} &= \mu Y_H + \mu_1 Y_{TH} - (d + \alpha) Y_a \\ \frac{dN}{dt} &= A - dN - \alpha Y_a \end{aligned} \tag{2}$$

The results of the equilibrium analysis are given in the following theorems;

THEOREM-There exists following four equilibria of the model discussed before.

$$(i) E_0 = \left(0, 0, 0, 0, \frac{A}{d} \right)$$

$$(ii) E_1 = (Y_T, 0, 0, 0, N) \text{ Which exists if } \beta_1 > d + v_1$$

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1)}{d\beta_1}, \bar{N} = \frac{A}{d}$$

where

$$(iii) E_2 = (0, \hat{Y}_H, \hat{Y}_a, 0, N), \text{ Which exists if } \beta_2 > d + \mu$$

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

where

(iv)The general case

$E_3 = (Y_T^*, Y_H^*, Y_{TH}^*, Y_a^*, N^*)$ is discussed below as follows;

$$Y_T^* = \frac{a_3 a_1 - a_1 b_2}{a_2 a_3 - a_1 a_4}, Y_H^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4 - a_2 a_3}$$

Where the constants are given by

$$\begin{aligned} a_1 &= \beta_1 \frac{\beta_1(\beta_1 - d - v_1)}{\beta_1(d + \mu - \beta_1(d + v_1))} - \frac{\beta_3 \beta_4 A(d + \alpha)}{\alpha\mu[\beta_1(d + \mu - \beta_1(d + v_1))] + \alpha\beta_2 d} - \frac{\beta_1 \beta_2 d}{\alpha\beta_2 d} \\ a_2 &= -\beta_1 - \beta_3 + \frac{\mu\beta_1}{\mu} + \frac{\beta_3 \beta_1(\beta_1 - d - v_1)}{[\beta_1(d + \mu - \beta_1(d + v_1))] + \alpha\mu[\beta_1(d + \mu - \beta_1(d + v_1))] + \alpha\beta_2 d} + \frac{d\beta_1 \beta_2}{\alpha\beta_2 d} \\ a_3 &= -\beta_1 - \beta_4 - \frac{\mu\beta_1}{\mu} - \frac{\beta_4 \beta_1(\beta_1 - d - \mu)}{[\beta_1(d + \mu - \beta_1(d + v_1))] + \alpha\mu[\beta_1(d + \mu - \beta_1(d + v_1))] + \alpha\beta_2 d} - \frac{d\beta_1 \beta_2}{\alpha\beta_2 d} \\ a_4 &= \frac{\beta_2 \beta_1(\beta_2 - d - \mu)}{[\beta_2(d + \mu - \beta_2(d + v_1))] + \alpha\mu[\beta_2(d + \mu - \beta_2(d + v_1))] + \alpha\beta_3 d} + \frac{\mu\beta_2}{\mu} + \frac{\beta_3 \beta_2 d(d + \alpha)}{\alpha\mu[\beta_2(d + \mu - \beta_2(d + v_1))] + \alpha\beta_3 d} + \frac{d\beta_2 \beta_3}{\alpha\beta_3 d} \end{aligned}$$

$$b_1 = \frac{A\beta_1}{\alpha} \left[1 + \frac{d + \alpha}{\mu_1} \right], \quad b_2 = \frac{A\beta_2}{\alpha} \left[1 + \frac{d + \alpha}{\mu_1} \right]$$

it will be +ive if

$$b_1 a_4 (a_2 b_2, a_1 a_4) a_2 a_3, a_1 b_2) a_3 b_1, a_2) 0, a_4) 0 \text{ i.e.}$$

$$\frac{\beta_2(d + v_1) - (d + \mu)0}{\beta_1} (\beta_1 - d - v_1) - \frac{\beta_1 A(\beta_2 - d - \mu)}{d\beta_2} - \frac{\beta_3 A(d + \alpha)(\beta_2 - d - \mu)}{d\beta_2(d + \alpha + \mu)} > 0$$

(FOR PROOF SEE APPENDIX—A)

4 STABILITY ANALYSIS

Now we study the local stability of equilibria

E_0, E_1, E_2 and E_3 and the non-linear

stability of non trivial equilibrium E_3 . The results are stated in the form of the following theorems.

THEOREM - 1

The equilibrium E_0, E_1, E_2 is unstable provided the following conditions are satisfied.

(i) $\beta_1 > d + v_1, \beta_2 > d + \mu$

(ii) $\frac{\beta_2(d + v_1)}{\beta_1} - (d + \mu) > 0$

(iii) $(\beta_1 - d - v_1) - \frac{\beta_1 A(\beta_2 - d - \mu)}{d\beta_2} - \frac{\beta_3 A(d + \alpha)(\beta_2 - d - \mu)}{d\beta_2(d + \alpha + \mu)} > 0$

(FOR PROOF SEE APPENDIX –B)

THEOREM -2

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$ is locally asymptotically stable provided that following conditions are satisfied.

(i) $[c_1(\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 < \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4N^{*2}}$

(ii) $c_1 \beta_1 Y_T^* < \frac{c_3 N^* (d + \alpha)}{4}$

(iii) $\left(\frac{c_1 \beta_1 Y_T^*}{N^*} - c_4 v_2 \right)^2 < \frac{c_1 c_4 (\theta + d) \beta_1 Y_T^*}{N^*}$

(iv) $c_1 Y_T^* (\beta_1 - d - v_1 - v_2)^2 < \frac{c_5 \beta_1 N^* d}{4}$

(v) $\left[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{4 N^*}$

(vi) $c_2 \beta_2 Y_H^* < \frac{c_4 (d + \theta) N^*}{4}$

(vii) $c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{c_5 d \beta_2 N^*}{4}$

(viii) $c_3 c_4 > 0$

(ix) $c_5 \alpha^2 < \frac{c_3 (d + \alpha)}{4}$

(x) $c_4 c_5 > 0$

[FOR PROOF SEE APPENDIX –C]

THEOREM-3

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$ is a non linear stable provided the following conditions are satisfied

(i) $(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{\beta_1 \beta_2 m_1}{4}$

(ii) $\left(\frac{\beta_2 m_1}{N^*} - \mu m_2 \right)^2 < \frac{\beta_1 m_2 (d + \alpha)}{4 N^*}$

(iii) $(4\beta_1 + \beta_3)^2 < \frac{d m_3 \beta_1 N^*}{4}$

(iv) $\left(\frac{-\beta_1}{N^*} + v_2 m_4 \right)^2 < \frac{m_4 \beta_1 (d + \theta)}{4 N^*}$

(v) $\left(\frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 < \frac{m_1 m_2 (d + \alpha) \beta_2}{4 N^*}$

(vi) $(4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} < \frac{d m_3 \beta_2}{4}$

(vii) $\left(\frac{\beta_2^2 m_1}{N^*} \right) < \frac{m_4 (d + \theta) \beta_2}{4}$

(viii) $m_3 \alpha^2 < \frac{m_2 d (d + \alpha)}{4}$

(ix) $m_2 m_4, m_3 m_4 > 0$

to prove the above theorem we need the following lema, the region of attraction is given by the following set.

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}, \quad 0 \leq Y_H \leq \frac{A(\beta_2 - d - \mu)}{d\beta_2}, \quad 0 \leq Y_a \leq \frac{\mu A(\beta_2 - d - \mu)}{d\beta_2(d + \alpha)}$$

$$0 \leq Z \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1(d + \theta)}, \quad 0 \leq N \leq \frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

A PARTICULAR CASE

SIS MODEL WITH CONSTANT IMMIGRATION

If in the model if we put $v_2 = 0, \theta=0$ it becomes sis model .

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + v_1 Y_T$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$$

where

$$N = X + Y_T + Y_H + Y_a$$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

all the symbols have their same meaning as in the previous model.

EQUILIBRIUM ANALYSIS

In this case there exists four equilibria of the model.

$$1; E_0 = \left(0, 0, 0, \frac{A}{d}\right)$$

$$2; E_1 = (Y_T, 0, 0, N) \quad \text{which exists if } \beta_1 \rangle d + v_1$$

where

$$Y_T = \frac{A(\beta_1 - d - v_1)}{d\beta_1}, \quad N = \frac{A}{d}$$

$$3; E_2 = (0, \hat{Y}_H, \hat{Y}_a, N) \quad \text{Which exists if } \beta_2 \rangle d + \mu$$

where

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$4; E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*), \quad \text{This is defined in three parts ;}$$

$$(a) E_{30} (Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, N_0^*) \quad \text{Where}$$

$$Y_{T0}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}, \quad Y_{H0}^* = \frac{b_2}{a_4}, \quad Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}, \quad N_0^* = \frac{A}{d} - \frac{\alpha}{d} \frac{\mu b_2}{a_4(\alpha + d)}$$

it exists only when

$$\beta_2 \rangle d + \mu$$

$$\left[\{\beta_1(d + \mu) - \beta_2(d + v_1)\} \left(1 + \frac{\mu}{\alpha + d}\right) - \beta_3(\beta_2 - d - \mu) \right] > 0$$

$E_{31}(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, N_1^*)$ Where

$$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}, \quad Y_{H1}^* = \frac{a_1 b_2 + a_2 b_1}{a_1 a_4 + a_2 a_3}, \quad Y_{a1}^* = \frac{\mu Y_{H1}^*}{d + \alpha}$$

$$N_1^* = \frac{A}{d} - \frac{\alpha \mu Y_{H1}^*}{d(d + \alpha)}, \quad \text{it exists provided the following conditions are satisfied.}$$

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{d + \alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$(b) \beta_1 \beta_2 \left[\frac{\alpha \mu}{d} + \alpha + d + \frac{\mu}{d + \alpha} \right] - \beta_3(\beta_2 - d - \mu) \frac{\alpha \mu}{d} - \mu(d + \alpha)\beta_2 > 0$$

$$\textcircled{c} E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*), \text{ (THE GENERAL CASE)}$$

Where

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, \quad Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, \quad Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$N_2^* = \frac{A}{d} - \frac{\alpha \mu Y_{H2}^*}{d(d + \alpha)}, \quad \text{it exists provided that}$$

$$a_4 b_1 \rangle a_2 b_2, \quad \text{and } a_1 b_2 \rangle a_3 b_1, \quad \text{i.e.}$$

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{d + \alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1 - v_2)}{\beta_1} - (d + \mu) > 0,$$

where

$$a_1 = \beta_1, \quad a_2 = -\frac{\alpha \mu(\beta_1 - d - v_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_2 \mu}{d + \alpha} + \beta_3, \quad a_3 = \beta_2 - \beta_3$$

$$a_4 = \frac{\alpha \mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu \beta_2}{d + \alpha}, \quad b_1 = \frac{A(\beta_1 - d - v_1)}{d}, \quad b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

(FOR PROOF SEE APPENDIX—A)

STABILITY ANALYSIS

now we study the local stability of equilibria

$$E_0, E_1, E_2 \quad \text{and } E_3 \quad \text{and the non-linear}$$

stability of non trivial equilibrium E_3 . The results are stated in the form of the following theorems.

THEOREM - 1;

The equilibrium E_0, E_1, E_2 is unstable provided the following conditions are satisfied.

$$(i) \beta_1 \rangle d + v_1, \quad \beta_2 \rangle d + \mu$$

$$(ii) \frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1)}{\beta_1} - (d + \mu) > 0$$

$$(iii) [\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{d + \alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

(FOR PROOF SEE APPENDIX –B)

THEOREM -2

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$ is locally asymptotically stable provided that following conditions are satisfied.

$$(i) [c_1(\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 < \frac{4c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{9N^{*2}}$$

$$(ii) c_1 \beta_1 Y_T^* < \frac{4c_3 N^* (d + \alpha)}{9}$$

$$(iii) c_1 Y_T^* (\beta_1 - d - v_1)^2 < \frac{4c_4 \beta_1 N^* d}{9}$$

$$(iv) \left[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{4c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{9N^*}$$

$$(v) \quad c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{4c_4 d \beta_2 N^*}{9}$$

$$(vi) \quad c_4 \alpha^2 < \frac{4c_3 (d + \alpha)}{9}$$

**[FOR PROOF SEE APPENDIX-C]
THEOREM-3**

The equilibrium $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$ is a non linearly stable provided the following conditions are satisfied

$$(i) \quad (\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{4\beta_1 \beta_2 m_1}{9}$$

$$(ii) \quad \beta_1 < \frac{4m_2 (d + \alpha) N^*}{9}$$

$$(iii) \quad (3\beta_1 + \beta_3)^2 < \frac{4dm_3 \beta_1 N^*}{9}$$

$$(iv) \quad \left(\frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 < \frac{4m_1 m_2 (d + \alpha) \beta_2}{9 N^*}$$

$$(v) \quad (\beta_2 m_1 + \beta_2 m_2 + \beta_3 m_1)^2 < \frac{4\beta_2 m_1 m_3 d N^*}{9}$$

$$(vi) \quad m_3 \alpha^2 < \frac{4m_2 d (d + \alpha)}{9}$$

To prove the above theorem we need the following lemma.

The region of attraction is

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1)}{d\beta_1}, 0 \leq Y_H \leq \frac{A(\beta_2 - d - \mu)}{d\beta_2}, 0 \leq Y_a \leq \frac{\mu A(\beta_3 - d - \mu)}{d\beta_3(d + \alpha)}, 0 \leq N \leq \frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

Conclusion

In this paper, a non linear model is proposed and analyzed to study the co- infection of TB and AIDS. It is shown that the spread of AIDS increases due to co-infection of TB and AIDS and the disease becomes endemic. It is also found that due to migration of HIV infection the spread of AIDS increases further.

APPENDIX-A

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H) X}{N} - dX + v_1 Y_T + \theta Z \quad (A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T - v_2 Y_T \quad (A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad (A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad (A_4)$$

$$\frac{dZ}{dt} = v_2 Y_T - dZ - \theta Z \quad (A_5)$$

Where $N = X + Y_T + Y_H + Y_a + Z$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A_6)$$

The reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1 + v_2) Y_T \dots \dots \dots (A_7)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_H - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \dots \dots \dots (A_8)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha) Y_a \dots \dots \dots (A_9)$$

$$\frac{dZ}{dt} = v_2 Y_T - dZ - \theta Z \dots \dots \dots (A_{10})$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a \dots \dots \dots (A_{11})$$

$$E_0 (0, 0, 0, 0, \frac{A}{d})$$

EXISTENCE OF

It is obvious from the model $(A_7 \dots \dots \dots A_{11})$,

we can find if $Y_T = Y_H = Y_a = 0$

$$N = \frac{A}{d}$$

So

This is disease free equilibrium.

EXISTENCE OF $E_1 (\bar{Y}_T, 0, 0, \bar{Z}, \bar{N})$

Since $Y_H = 0 \Rightarrow Y_a = 0$ and $Y_T \neq 0$
From

$$(A_{11}) \text{ we get } \bar{N} = \frac{A}{d}, \quad \text{From}$$

$$(A_{10}), \text{ we get } Z = \frac{v_2 Y_T}{\theta + d}$$

from (A_7) we get

$$\beta_1 Y_T (\bar{N} - \bar{Y}_T - \bar{Z}) - (d + v_1 + v_2) \bar{Y}_T \bar{N} = 0$$

Since $\bar{Y}_T \neq 0$

So on solving, we get

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1 v}, \bar{Z} = \frac{v_2 Y_T}{d + \theta}, \bar{N} = \frac{A}{d}$$

Which exists if $\beta_1 > d + v_1 + v_2$

EXISTENCE OF $E_2 (0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$

Since $\hat{Y}_T = 0 \Rightarrow \hat{Z} = 0$, And $\hat{Y}_H, \hat{Y}_a \neq 0$

So, on solving A_8, A_9, A_{11} We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

Which exists if $\beta_2 > d + \mu$

EXISTENCE OF $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$

From (A_9) , we get $Y_a^* = \frac{\mu Y_H^*}{d + \alpha}$, from (A_{10}) We get $Z^* = \frac{v_2 Y_T^*}{d + \theta}$

From (A_{11}) $N^* = \frac{A}{d} - \frac{\alpha \mu Y_H^*}{d(d + \alpha)}$

So A_7 and A_8 Becomes

$$\beta_1 Y_T^* + Y_H^* \left[\frac{\alpha \mu (\beta_2 - d - v_1 - v_2)}{d(d + \alpha)} + \beta_1 + \frac{\beta_2 \mu}{d + \alpha} + \beta_3 \right] = \frac{A \beta_2 (d - v_1 - v_2)}{d} \dots (A_2)$$

$$v_1 (\beta_1 v_1 - \beta_1) + v_2 \left[\frac{\alpha \mu (\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_1 + \frac{\beta_2 \mu}{d + \alpha} \right] = \frac{A \mu (\beta_2 - d - \mu)}{d} \dots (A_3)$$

where

$$a_1 = v_1 \beta_1, a_2 = \frac{\alpha \mu (\beta_2 - d - v_1 - v_2)}{d(d + \alpha)} + \beta_1 + \frac{\beta_2 \mu}{d + \alpha} + \beta_3, a_3 = \beta_2 v_1 - \beta_3$$

$$a_4 = \frac{\alpha \mu (\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\beta_2 \mu}{d + \alpha}, b_1 = \frac{A \mu (\beta_2 - d - v_1 - v_2)}{d}, b_2 = \frac{A (\beta_2 - d - \mu)}{d}$$

$$v = 1 + \frac{v_2}{d + \theta}$$

so A_{12} And A_{13} Becomes

$$a_4 Y_T^* + a_5 Y_H^* = b_1, a_1 Y_T^* + a_2 Y_H^* = b_2 \dots (A_4)$$

now here arise the three cases –

(1); if $a_3 = 0 \Rightarrow \beta_2 v = \beta_3$ so the equilibrium $E_{30} (Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, Z_0, N_0^*)$

so

$$Y_{T0}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4 (\alpha + d)}$$

$$Z_0^* = \frac{v_2 (b_1 a_4 - a_2 b_2)}{a_1 a_4}, N_0^* = \frac{A}{d} - \frac{\alpha}{d} \frac{\mu b_2}{a_4 (\alpha + d)}$$

it exists only when

$$\beta_2 > d + \mu$$

$$\left[v \beta_1 (d + \mu) - \beta_2 (d + v_1 + v_2) \right] \left(1 + \frac{\mu}{d + \alpha} \right) - \beta_3 (\beta_2 - d - \mu) > 0$$

(2) if < 0 so The Equilibrium b) $E_{31} (Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, Z_1^*, N_1^*)$ Where

$$Y_{T1}^* = \frac{a_2 b_1 - a_1 b_2}{a_1 a_4 + a_2 a_3}, Y_{H1}^* = \frac{a_1 b_2 + a_2 b_1}{a_1 a_4 + a_2 a_3}, Z_1^* = \frac{v_2 Y_{T1}^*}{d + \theta}$$

$N_1^* = \frac{A}{d} - \frac{\alpha \mu Y_{H1}^*}{d(d + \alpha)}$, it exists provided the following conditions are satisfied.

$$[v \beta_1 (d + \mu) - \beta_2 (d + v_1 + v_2) \left(1 + \frac{\mu}{d + \alpha} \right) - \beta_3 (\beta_2 - d - \mu)] > 0$$

$$v \beta_1 \beta_2 \left[\frac{\alpha \mu}{d} + \alpha + d + \frac{\mu}{d + \alpha} \right] - \beta_3 (\beta_2 - d - \mu) \frac{\alpha \mu}{d} - \mu (d + \alpha) \beta_2 > 0$$

(3) if $a_3 > 0$ so The Equilibrium ©

$$E_{32} (Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*), (THE GENERAL CASE)$$

Where

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$Z_2^* = \frac{v_2 Y_{T2}^*}{d + \theta}, N_2^* = \frac{A}{d} - \frac{\alpha \mu Y_{H2}^*}{d(d + \alpha)}, \text{ it exists provided that}$$

$a_1 b_1 > a_2 b_2$, and $a_1 b_2 > a_3 b_1$, i.e.

$$[v \beta_1 (d + \mu) - \beta_2 (d + v_1 + v_2) \left(1 + \frac{\mu}{d + \alpha} \right) - \beta_3 (\beta_2 - d - \mu)] > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3 (\beta_1 - d - v_1 - v_2)}{\beta_1} - (d + \mu) v > 0,$$

APPENDIX-B

Using the model (A_7, \dots, A_{11}) , The general jacobian matrix can be written as follows;

$$M = \begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ 0 & \mu & -(\alpha + d) & 0 & 0 \\ v_2 & 0 & 0 & -(\theta + d) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

where

$$m_{11} = \frac{\beta_1 (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - v_1 - v_2$$

$$m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}, m_{14} = \frac{-\beta_1 Y_T}{N}$$

$$m_{15} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2 (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d + v)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}, m_{24} = \frac{-\beta_2 Y_H}{N}$$

$$m_{25} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_a - Z)}{N^2} -$$

$$\frac{\beta_3 Y_T Y_H}{N^2}$$

PROOF OF THEOREM-1 stability of

$$E_0 (0, 0, 0, 0, \frac{A}{d})$$

The jacobian matrix for E_0 is given by

$$M(E_0) = \begin{matrix} \beta_1 - d - v_1 - v_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 - d - \mu & 0 & 0 & 0 \\ 0 & \mu & -(d + \alpha) & 0 & 0 \\ v_2 & 0 & 0 & -(d + \theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

the eigen values are

$$\beta_1 - d - v_1 - v_2, \beta_2 - d - \mu, -(d + \alpha), -(d + \theta), -d$$

since $\beta_1 > d + v_1 + v_2$, $\beta_2 > d + \mu$ so

the equilibrium E_0 is unstable.

Stability for $E_1 (\bar{Y}_T, 0, 0, \bar{Z}, \bar{N})$

The jacobian matrix for E_1 is given as

$$M(E_1) = \begin{matrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & \mu & -(\alpha + d) & 0 & 0 \\ v_2 & 0 & 0 & -(d + \theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

where

$$b_{11} = (\beta_1 - d - v_1 - v_2), b_{12} = -v_1 \beta_1 (\beta_1 - d - v_1 - v_2), b_{13} = b_{14} = -(\beta_1 - d - v_1 - v_2)$$

$$b_{15} = \frac{(\beta_1 - d - v_1 - v_2)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1}d + \frac{\beta_3}{\beta_1}$$

$$(\beta_1 - d - v_1 - v_2) - v(d + \mu)$$

Since eigen value are

$$b_{11}, b_{22}, -(\alpha + d), -(\theta + d), -d$$

Since $b_{22} > 0$ so E_1 is unstable

stability of $E_2(0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$,

The jacobian for E_2 is

$$M(E_2) = \begin{pmatrix} \hat{a}_1 & 0 & 0 & 0 & 0 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 & \hat{b}_5 \\ 0 & \mu & -(\alpha + d) & 0 & 0 \\ v_2 & 0 & 0 & -(\theta + d) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{pmatrix}$$

Where

$$\hat{a}_1 = (\mu + d) \frac{\beta_1(\beta + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta \frac{\beta_2(\alpha + d)(\beta_2 - d - \mu)}{\alpha + d + \mu}, \hat{b}_1 = \frac{\beta - \beta_2(\alpha + d)(\beta_2 - d - \mu)}{\beta_1}$$

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu}$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}, \hat{b}_4 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is \hat{a}_1 which is > 0 i.e.

E_2 is unstable

APPENDIX-C

Local stability of Equilibrium

$E_3(Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$ By Lyapunov method

The model $(A_7 - A_{11})$ can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, Y_H = Y_H^* + y_H, Y_a = Y_a^* + y_a, Z = Z^* + z, N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}z + m_{15}n$$

$$\frac{dY_H}{dt} = m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}z + m_{25}n$$

$$\frac{dy_a}{dt} = \mu y_H - (d + \alpha)y_a$$

$$\frac{dz}{dt} = v_2 y_T - (d + \theta)z$$

$$\frac{dn}{dt} = -\alpha Y_a - dn$$

Where we use following Lyapunov function;

Where C_1, C_2, C_3, C_4, C_5 are to be determined.

So

$$\dot{V} = c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 z \frac{dz}{dt} + c_5 n \frac{dn}{dt}$$

$$= c_1 y_T (m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}z + m_{15}n) + c_2 y_H (m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}z + m_{25}n)$$

+

$$c_3 y_a (\mu y_H - (d + \alpha)y_a) + c_4 z (v_2 y_T - (d + \theta)z) + c_5 n (-\alpha y_a - dn)$$

$$= \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T y_H (c_1 m_{12} + c_2 m_{21}) + \frac{1}{4} c_1 m_{13} y_a^2 \right] + \left[\frac{1}{4} c_2 m_{22} y_H^2 + y_T y_H (c_1 m_{12} + c_2 m_{21}) - \frac{1}{4} c_2 (d + \alpha) y_a^2 \right]$$

$$+ \left[\frac{1}{4} c_3 m_{23} y_a^2 + y_T z (c_1 m_{14} + c_2 m_{24}) - \frac{1}{4} c_4 (d + \theta) z^2 \right] + \left[\frac{1}{4} c_4 m_{14} y_T^2 + y_T n (c_1 m_{15} + c_2 m_{25}) - \frac{1}{4} c_5 d n^2 \right]$$

$$+ \left[\frac{1}{4} c_5 m_{25} y_a^2 + y_a n (c_1 m_{15} + c_2 m_{25}) - \frac{1}{4} c_5 (d + \alpha) y_a^2 \right] + \left[\frac{1}{4} c_1 m_{11} y_T^2 + y_T n (c_1 m_{15} + c_2 m_{25}) - \frac{1}{4} c_1 (d + \theta) z^2 \right]$$

$$+ \left[\frac{1}{4} c_2 m_{22} y_H^2 + y_H n (c_2 m_{25}) - \frac{1}{4} c_2 d n^2 \right] + \left[-\frac{1}{4} c_1 (d + \alpha) y_a^2 \right] + \left[-\frac{1}{4} c_2 (d + \theta) z^2 + 0 - \frac{1}{4} c_5 d n^2 \right]$$

$$+ \left[-\frac{1}{4} c_1 (d + \alpha) y_a^2 + y_a n (c_1 m_{15} + c_2 m_{25}) - \frac{1}{4} c_1 d n^2 \right] + \left[-\frac{1}{4} c_2 (d + \theta) z^2 + 0 - \frac{1}{4} c_5 d n^2 \right]$$

for the equilibrium $E_3(Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$

$$m_{11} = \frac{-\beta Y_T^*}{N^*}, m_{12} = (-\beta - \beta_1) \frac{Y_T^*}{N^*}, m_{13} = \frac{-\beta Y_a^*}{N^*}, m_{14} = m_{15} = (\beta - d - v_1 - v_2) \frac{Y_T^*}{N^*}$$

$$m_{21} = (-\beta_2 + \beta_1) \frac{Y_T^*}{N^*}, m_{22} = m_{23} = m_{24} = \frac{\beta Y_H^*}{N^*}, m_{25} = ((\beta_2 - d - \mu) \frac{Y_T^*}{N^*})$$

since all the coefficients of square terms are negative so V will be negative definite provided the following conditions are satisfied.

$$(i) [c_1(\beta_1 + \beta_2) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 < \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4 N^{*2}}$$

$$(ii) c_1 \beta_1 Y_T^* < \frac{c_3 N^* (d + \alpha)}{4}$$

$$(iii) \left(\frac{c_1 \beta_1 Y_T^*}{N^*} - c_4 v_2 \right)^2 < \frac{c_1 c_4 (\theta + d) \beta_1 Y_T^*}{N^*}$$

$$(iv) c_1 Y_T^* (\beta_1 - d - v_1 - v_2)^2 < \frac{c_5 \beta_1 N^{*2}}{4}$$

$$(v) \left[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{4 N^{*2}}$$

$$(vi) c_2 \beta_2 Y_H^* < \frac{c_4 (d + \theta) N^*}{4}$$

$$(vii) c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{c_5 d \beta_2 N^*}{4}$$

$$(viii) c_3 c_4 > 0$$

$$(ix) c_5 \alpha^2 < \frac{c_3 (d + \alpha)}{4}$$

$$(x) c_4 c_5 > 0$$

APPENDIX-D (NON LINEAR STABILITY) PROOF OF THE LEEMA

From model $(A_7 \dots A_9)$ we can obtain as follows;

$$\frac{dN}{dt} = A - dN - \alpha Y_a \leq A - dN \Rightarrow N \leq \frac{A}{d}$$

$$\frac{dY_T}{dt} = \frac{\beta Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta Y_T Y_H}{N} - (d + v_1 + v_2) Y_T$$

$$\leq (\beta_1 - d - v_1 - v_2) Y_T - \frac{\beta_1 Y_T^2}{N}$$

$$\Rightarrow Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}$$

$$Y_H \leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$$

similarly

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$$

$$\Rightarrow Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d(d + \alpha)}$$

$$\frac{dZ}{dt} = v_2 Y_T - dZ - \theta Z \Rightarrow Z \leq \frac{A(\beta_1 - d - v_1 - v_2)v_2}{d\beta_1(d + \theta)}$$

so the region of attraction is

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}$$

$$0 \leq Y_H \leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$$

$$0 \leq Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d(d + \alpha)}, \quad 0 \leq Z \leq \frac{A(\beta_1 - d - v_1 - v_2)v_2}{d\beta_1(d + \theta)}$$

$$0 \leq N \leq \frac{A}{d}$$

PROOF OF THE THEOREM

In this case we use the following Lyapunov

function;

$$W = \left[Y_T - Y_T^* \log \frac{Y_T}{Y_T^*} \right] + m \left[Y_H - Y_H^* \log \frac{Y_H}{Y_H^*} \right] + \frac{m_2}{2} (Y_T - Y_T^*)^2 + \frac{m_2}{2} (N - N^*)^2 + \frac{m_2}{2} (Z - Z^*)^2$$

on differentiating and substituting the values from model we get

$$\begin{aligned} & \left[\frac{-\beta_1}{4N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_H - Y_H^*) \left(\frac{-\beta_1 - \beta_2 - \beta_2 m_1 + \beta_2 m_1}{N^*} \right) + \frac{1}{4} (Y_T - Y_T^*) \left(\frac{-\beta_2 m_1}{N^*} \right) \right] \\ & + \left[\frac{-\beta_2}{4N^*} (Y_H - Y_H^*)^2 + (Y_T - Y_T^*) (Y_H - Y_H^*) \left(\frac{-\beta_2 m_1}{N^*} + \mu m_1 \right) - \frac{1}{4} m_2 (d + \alpha) (Y_T - Y_T^*)^2 \right] \\ & + \left[\frac{-\beta_2}{4N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (N - N^*) \left(\frac{\beta_2 Y_T + \beta_2 Z + \beta_2 Y_a + (\beta_2 + \beta_2) Y_H}{NN^*} \right) - \frac{1}{4} d m_2 (N - N^*)^2 \right] \\ & + \left[\frac{-\beta_2}{4N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Z - Z^*) \left(\frac{-\beta_2}{N^*} + v_2 m_1 \right) - \frac{1}{4} m_2 (d + \theta) (Z - Z^*)^2 \right] \\ & + \left[\frac{1}{4} (Y_T - Y_T^*)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) + (Y_T - Y_T^*) (Y_H - Y_H^*) \left(\frac{-\beta_2 m_1}{N^*} + \mu m_1 \right) - \frac{1}{4} m_2 (d + \alpha) (Y_T - Y_T^*)^2 \right] \\ & + \left[\frac{1}{4} (Y_H - Y_H^*)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) + (Y_H - Y_H^*) (N - N^*) \left(\frac{\beta_2 Y_T + \beta_2 Z + \beta_2 Y_a + (\beta_2 + \beta_2) Y_H}{NN^*} \right) - \frac{1}{4} d m_2 (N - N^*)^2 \right] \\ & + \left[\frac{1}{4} (Y_T - Y_T^*)^2 \left(\frac{-\beta_2 m_1}{N^*} \right) + (Y_T - Y_T^*) (Z - Z^*) \left(\frac{-\beta_2}{N^*} + v_2 m_1 \right) - \frac{1}{4} m_2 (d + \theta) (Z - Z^*)^2 \right] \\ & + \left[-\frac{1}{4} m_2 (d + \alpha) (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (N - N^*) \left(\frac{\beta_2 Y_T + \beta_2 Z + \beta_2 Y_a + (\beta_2 + \beta_2) Y_H}{NN^*} \right) - \frac{1}{4} d m_2 (N - N^*)^2 \right] \\ & + \left[-\frac{1}{4} m_2 (d + \alpha) (Y_T - Y_T^*)^2 + 0 - \frac{1}{4} d m_2 (d + \theta) (Z - Z^*)^2 \right] + \left[-\frac{1}{4} d m_2 (N - N^*)^2 + 0 - \frac{1}{4} d m_2 (N - N^*)^2 \right] \end{aligned}$$

$$\frac{Y_T}{N}, \frac{Y_H}{N}, \frac{Y_a}{N}, \frac{Z}{N} = 1$$

CONSIDER

W is negative definite if the following conditions are satisfied ;

$$(i) (\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{\beta_1 \beta_2 m_1}{4}$$

$$(ii) \left(\frac{\beta_2 m_1}{N^*} - \mu m_2 \right)^2 < \frac{\beta_1 m_2 (d + \alpha)}{4 N^*}$$

$$(iii) (4\beta_1 + \beta_3)^2 < \frac{d m_3 \beta_1 N^*}{4}$$

$$(iv) \left(\frac{-\beta_1}{N^*} + v_2 m_4 \right)^2 < \frac{m_4 \beta_1 (d + \theta)}{4 N^*}$$

$$(v) \left(\frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 < \frac{m_1 m_2 (d + \alpha) \beta_2}{4 N^*}$$

$$(vi) (4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} < \frac{d m_3 \beta_2}{4}$$

$$(vii) \left(\frac{\beta_2 m_1}{N^*} \right)^2 < \frac{m_4 (d + \theta) \beta_2}{4}$$

$$(viii) m_3 \alpha^2 < \frac{m_2 d (d + \alpha)}{4}$$

$$(ix) m_2 m_4, m_3 m_4 > 0$$

FOR SIS MODEL;
APPENDIX-A1

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H) X}{N} - dX + v_1 Y_T \quad (A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T \quad (A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad (A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad (A_4)$$

$$\text{Where } N = X + Y_T + Y_H + Y_a$$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A_5)$$

the reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1) Y_T \dots \dots \dots (A_6)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \dots \dots \dots (A_7)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha) Y_a \dots \dots \dots (A_8)$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a \dots \dots \dots (A_9)$$

$$\text{EXISTENCE OF } E_0(0,0,0, \frac{A}{d})$$

It is obvious from the model (A_6, \dots, A_9) , we can find if $Y_T = Y_H = Y_a = 0$

So
$$N = \frac{A}{d}$$

This is disease free equilibrium.

EXISTENCE OF $E_1(\bar{Y}_T, 0, 0, \bar{N})$

Since $Y_H = 0 \Rightarrow Y_a = 0$ and $Y_T \neq 0$

From (A_9) we get $\bar{N} = \frac{A}{d}$,

from (A_6) we get

$$\beta_1 Y_T (\bar{N} - \bar{Y}_T) - (d + v_1) \bar{Y}_T \bar{N} = 0$$

Since $\bar{Y}_T \neq 0$

So on solving, we get

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1 v}, \bar{N} = \frac{A}{d}$$

Which exists if $\beta_1 > d + v_1$

EXISTENCE OF $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$

Since $\hat{Y}_T = 0$ And $\hat{Y}_H, \hat{Y}_a \neq 0$

So, on solving A_7, A_8, A_9 We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}, \text{ Which exists if}$$

$\beta_2 > d + \mu$

EXISTENCE OF $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$

From

(A_8) , we get $Y_a^* = \frac{\mu Y_T^*}{d + \alpha}$,

From (A_9) $N^* = \frac{A}{d} - \frac{\alpha\mu Y_H^*}{d(\alpha + d)}$

So A_6 and A_7 Becomes

$$\beta_1 Y_T + Y_H \left[\frac{\alpha\mu\beta_1 - d - v_1}{d(\alpha + d)} + \beta_1 + \frac{\beta_2\mu}{d + \alpha} + \beta_3 \right] = \frac{A(\beta_1 - d - v_1)}{d} \dots (A_0)$$

$$Y_T(\beta_1 - \beta_2) + Y_H \left[\frac{\alpha\mu\beta_2 - d - \mu}{d(\alpha + d)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha} \right] = \frac{A(\beta_2 - d - \mu)}{d} \dots (A_1)$$

where

$$a_1 = \beta_1, a_2 = \frac{\alpha\mu(\beta_1 - d - v_1)}{d(\alpha + d)} + \beta_1 + \frac{\beta_2\mu}{d + \alpha} + \beta_3, a_3 = \beta_2 - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(\alpha + d)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha}, b_1 = \frac{A(\beta_1 - d - v_1)}{d}, b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

so A_{12} And A_{13} Becomes

$$a_1 Y_T^* + a_2 Y_H^* = b_1, a_3 Y_T^* + a_4 Y_H^* = b_2 \dots (A_2)$$

now here arise the three cases –

(1) if $a_3 = 0 \Rightarrow \beta_2 = \beta_3$ so the equilibrium $E_{30}(Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, N_0^*)$

so

$$Y_{T0}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}$$

$$Z_0^* = \frac{v_2(b_1 a_4 - a_2 b_2)}{a_1 a_4}, N_0^* = \frac{A}{d} - \frac{\alpha}{d} \frac{\mu b_2}{a_4(\alpha + d)}$$

it exists only when

$$\beta_2 > d + \mu$$

$$\left[\{\beta_1(d + \mu) - \beta_2(d + v_1)\} \left(1 + \frac{\mu}{\alpha + d}\right) - \beta_3(\beta_2 - d - \mu) \right] > 0$$

(2) if $a_3 < 0$ so The Equilibrium $E_{31}(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, N_1^*)$. Where

$$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}, Y_{H1}^* = \frac{a_4 b_2 + a_2 b_1}{a_1 a_4 + a_2 a_3}$$

$$N_1^* = \frac{A}{d} - \frac{\alpha\mu Y_{H1}^*}{d(\alpha + d)}, \text{ it exists provided the following conditions are satisfied.}$$

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{\alpha + d}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\beta_1 \beta_2 \left[\frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d + \alpha} \right] - \beta_3(\beta_2 - d - \mu) \frac{\alpha\mu}{d} - \mu(d + \alpha)\beta_2 > 0$$

(3) if $a_3 > 0$ so The Equilibrium \odot

$E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, N_2^*)$, (THE GENERAL CASE)

Where

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_4 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$N_2^* = \frac{A}{d} - \frac{\alpha\mu Y_{H2}^*}{d(\alpha + d)}, \text{ it exists provided that}$$

$a_4 b_1 > a_2 b_2$, and $a_4 b_2 > a_3 b_1$, i.e.

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{\alpha + d}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1)}{\beta_1} - (d + \mu) > 0$$

APPENDIX-B1

Using the model (A_6, \dots, A_9) , The general jacobian matrix can be written as follows;

$$M = \begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \mu & -(\alpha + d) & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & -\alpha & -d \end{matrix}$$

where

$$m_{11} = \frac{\beta_1(N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - v_1$$

$$m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}$$

$$m_{14} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T(N - Y_T - Y_H - Y_a)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2(N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d + \mu)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}$$

$$m_{24} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_a)}{N^2} - \frac{\beta_3 Y_T Y_H}{N^2}$$

PROOF OF THEOREM-1; stability of $E_0(0,0,0, \frac{A}{d})$

The jacobian matrix for E_0 is given by

$$M(E_0) = \begin{pmatrix} \beta_1 - d - v_1 - v_2 & 0 & 0 & 0 \\ 0 & \beta_2 - d - \mu & 0 & 0 \\ 0 & \mu & -(d + \alpha) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

the eigen values are $\beta_1 - d - v_1, \beta_2 - d - \mu, -(d + \alpha), -(d + \theta)$,
since $\beta_1 > d + v_1, \beta_2 > d + \mu$ so

the equilibrium E_0 is unstable.

Stability for $E_1(\bar{Y}_T, 0, 0, \bar{N})$

The jacobian matrix for E_1 is given as

$$M(E_1) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & 0 & 0 \\ 0 & \mu & -(d + \alpha) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

where

$$b_{11} = -(\beta_1 - d - v_1), b_{12} = -(\beta_1 + \beta_3)(\beta_1 - d - v_1), b_{13} = -(\beta_1 - d - v_1)$$

$$b_{14} = \frac{(\beta_1 - d - v_1)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1} d + \frac{\beta_3}{\beta_1} (\beta_1 - d - v_1) - (d + \mu)$$

Since eigen value are

$$b_{11}, b_{22}, -(d + \alpha), -(d + \theta)$$

Since $b_{22} > 0$ so E_1 is unstable

stability of $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$

The jacobian for E_2 is

$$M(E_2) = \begin{pmatrix} \hat{a}_1 & 0 & 0 & 0 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 \\ 0 & \mu & -(d + \alpha) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

Where

$$\hat{a}_1 = (\mu + d)\beta_1 - \frac{\beta_1(\beta_1 + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta_1(d + v_1), \hat{b}_1 = \frac{\beta_1 - \beta_2}{\beta_2}, \hat{b}_2 = \frac{(\alpha + d)(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu}$$

$$b_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}, \hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is a_1 which is >0 i.e. E_2 is unstable

APPENDIX-C1

Local stability of Equilibrium $E_3(Y_T^*, Y_H^*, Y_a^*, N^*)$ By Lyapunov method,

The model $(A_6 - A_9)$ can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, Y_H = Y_H^* + y_H, Y_a = Y_a^* + y_a, N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}n$$

$$\frac{dY_H}{dt} = m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}n$$

$$\frac{dy_a}{dt} = \mu y_H - (d + \alpha)y_a$$

$$\frac{dn}{dt} = -\alpha y_a - dn$$

Where we use following Lyapunov function ;

$$V = \frac{c_1 y_T^2}{2} + \frac{c_2 y_H^2}{2} + \frac{c_3 y_a^2}{2} + \frac{c_4 n^2}{2}$$

Where c_1, c_2, c_3, c_4 are to be determined.

So

$$\begin{aligned} \dot{V} &= c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 n \frac{dn}{dt} \\ &= c_1 y_T (m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}n) + c_2 y_H (m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}n) \\ &+ c_3 y_a (\mu y_H - (d + \alpha)y_a) + c_4 n (-\alpha y_a - dn) \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{1}{3} c_1 m_1 Y_T^2 + Y_T Y_H (c_1 m_2) + \frac{1}{3} c_2 m_2 Y_H^2 \right] + \left[\frac{1}{4} c_1 m_1 Y_T^2 + Y_T Y_H (c_1 m_3) - \frac{1}{4} c_1 (d + \alpha) Y_H^2 \right] \\
 &+ \left[\frac{1}{3} c_1 m_1 Y_T^2 + Y_T Y_H (c_1 m_4) - \frac{1}{3} c_1 d N^2 \right] \\
 &+ \left[\frac{1}{3} c_2 m_2 Y_H^2 + Y_H Y_H (c_2 m_2 + \mu c_3) - \frac{1}{4} c_2 (d + \alpha) Y_H^2 \right] \\
 &+ \left[\frac{1}{3} c_2 m_2 Y_H^2 + Y_H Y_H (c_2 m_2) - \frac{1}{3} c_2 d N^2 \right] \\
 &+ \left[-\frac{1}{3} c_3 (d + \alpha) Y_H^2 + Y_H Y_H (-c_3 \alpha) - \frac{1}{3} c_3 d N^2 \right] \\
 &\text{for the equilibrium } E_3(Y_T^*, Y_H^*, Y_a^*, N^*) \\
 m_{11} &= \frac{-\beta_1 Y_T^*}{N^*}, m_{12} = (-\beta_1 - \beta_2) \frac{Y_T^*}{N^*}, m_{13} = \frac{-\beta_1 Y_T^*}{N^*}, m_{14} = (\beta_1 - d - v_1) \frac{Y_T^*}{N^*} \\
 m_{21} &= (-\beta_2 + \beta_1) \frac{Y_H^*}{N^*}, m_{22} = m_{23} = \frac{\beta_2 Y_H^*}{N^*}, m_{24} = ((\beta_2 - d - \mu) \frac{Y_H^*}{N^*}
 \end{aligned}$$

since all the coefficients of square terms are negative so \dot{V} WILL be negative definite provided the following conditions are satisfied.

$$\begin{aligned}
 \text{(i)} \quad & [c_1(\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 < \frac{4c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{9N^{*2}} \\
 \text{(ii)} \quad & c_1 \beta_1 Y_T^* < \frac{4c_3 N^* (d + \alpha)}{9} \\
 \text{(iii)} \quad & c_1 Y_T^* (\beta_1 - d - v_1)^2 < \frac{4c_4 \beta_1 N^* d}{9} \\
 \text{(iv)} \quad & \left[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{4c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{9N^{*2}} \\
 \text{(v)} \quad & c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{4c_4 d \beta_2 N^*}{9} \\
 \text{(vi)} \quad & c_4 \alpha^2 < \frac{4c_3 d (d + \alpha)}{9}
 \end{aligned}$$

**APPENDIX-D1
(NON LINEAR STABILITY)
PROOF OF THE LEEMA**

From model $(A_6 \dots A_9)$ we can obtain as follows;

$$\begin{aligned}
 \frac{dN}{dt} &= A - dN - \alpha Y_a \leq A - dN \Rightarrow N \leq \frac{A}{d} \\
 \frac{dY_T}{dt} &= \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_2 Y_T Y_H}{N} - (d + v_1) Y_T \\
 &\leq (\beta_1 - d - v_1) Y_T - \frac{\beta_1 Y_T^2}{N} \\
 \Rightarrow Y_T &\leq \frac{A(\beta_1 - d - v_1)}{d\beta_1} \\
 Y_H &\leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}
 \end{aligned}$$

similarly and

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \Rightarrow Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d (d + \alpha)}$$

so the region of attraction is

$$\begin{aligned}
 0 &\leq Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}, \\
 0 &\leq Y_H \leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}, \\
 0 &\leq Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d (d + \alpha)}, \\
 0 &\leq N \leq \frac{A}{d}
 \end{aligned}$$

PROOF OF THE THEOREM

In this case we use the following Lyapunov function;

$$\begin{aligned}
 W &= (Y_T - Y_T^* - Y_T^* \log \frac{Y_T}{Y_T^*}) + m_1 (Y_H - Y_H^* - Y_H^* \log \frac{Y_H}{Y_H^*}) + \frac{m_2}{2} (Y_a - Y_a^*)^2 + \frac{m_3}{2} (N - N^*)^2 \\
 &\text{on differentiating and substituting the values from model we get:} \\
 \dot{W} &= \left[\frac{-\beta_1}{3N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_H - Y_H^*) \left\{ \frac{-\beta_1 - \beta_2 - \beta_3 m_1 + \beta_3 m_2}{N^*} \right\} + \frac{1}{3} (Y_H - Y_H^*) \left\{ \frac{-\beta_2 m_1}{N^*} \right\} \right] \\
 &+ \left[\frac{-\beta_2}{3N^*} (Y_H - Y_H^*)^2 + (Y_H - Y_H^*) (Y_a - Y_a^*) \left\{ \frac{-\beta_2 m_1}{N^*} \right\} - \frac{1}{4} m_2 (d + \alpha) (Y_a - Y_a^*)^2 \right] \\
 &+ \left[\frac{-\beta_3}{3N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (N - N^*) \left\{ \frac{\beta_1 Y_T^* + \beta_2 Y_H^* + (\beta_1 + \beta_2) Y_a^*}{NN^*} \right\} - \frac{1}{3} dm_3 (N - N^*)^2 \right] \\
 &+ \left[\frac{1}{3} (Y_H - Y_H^*)^2 \left\{ \frac{-\beta_2 m_1}{N^*} \right\} + (Y_H - Y_H^*) (Y_a - Y_a^*) \left\{ \frac{-\beta_2 m_1}{N^*} + \mu m_2 \right\} - \frac{1}{3} m_2 (d + \alpha) (Y_a - Y_a^*)^2 \right] \\
 &+ \left[\frac{1}{3} (Y_H - Y_H^*)^2 \left\{ \frac{-\beta_2 m_1}{N^*} \right\} + (Y_H - Y_H^*) (N - N^*) \left\{ \frac{\beta_2 m_1 Y_T^* + (Y_H - Y_H^*) \beta_2 m_1 - \beta_2 m_1 Y_a^*}{NN^*} \right\} - \frac{1}{3} dm_3 (N - N^*)^2 \right] \\
 &+ \left[-\frac{1}{3} m_2 (d + \alpha) (Y_a - Y_a^*)^2 + (Y_a - Y_a^*) (N - N^*) \left\{ -m_2 \alpha \right\} - \frac{1}{3} dm_3 (N - N^*)^2 \right]
 \end{aligned}$$

$$\frac{Y_T}{N}, \frac{Y_H}{N}, \frac{Y_a}{N} = 1$$

CONSIDER \dot{W} is negative definite if the following conditions are satisfied ;

$$\begin{aligned}
 \text{(i)} \quad & (\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_2)^2 < \frac{4\beta_1 \beta_2 m_1}{9} \\
 \text{(ii)} \quad & \beta_1 < \frac{4m_2 (d + \alpha) N^*}{9} \\
 \text{(iii)} \quad & (3\beta_1 + \beta_3)^2 < \frac{4dm_3 \beta_1 N^*}{9} \\
 \text{(iv)} \quad & \left(\frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 < \frac{4m_1 m_2 (d + \alpha) \beta_2}{9N^{*2}} \\
 \text{(v)} \quad & (\beta_2 m_1 + \beta_2 m_2 + \beta_3 m_1)^2 < \frac{4\beta_2 m_1 m_3 d N^*}{9} \\
 \text{(vi)} \quad & m_3 \alpha^2 < \frac{4m_2 d (d + \alpha)}{9}
 \end{aligned}$$

References

- [1] Fuzzi, et al. (1996) *Journal of Biological Mathematics*, 37, p 371-375.
- [2] Dufar (1982) *Journal of Biological Mathematics*, 37, 391-393.
- [3] Waltman and Hethcote(1974) *Journal of Biological Mathematics*, 39, 381-398.
- [4] Shukla J.B. (1987) *Journal of Biological Mathematics*, 23, 121-131.

- [5] [Cooke,1979,Marc and Hethcote,1976Gonzalez,Guzman,1989] *Journal of Biological Mathematics*
- [6] Ramkrishnan and Chandrashekhar (1999) *Journal of Biological Mathematics*, p 232-238.
- [7] May Anderson (1987) *Journal of Biological Mathematics*, 54-65.
- [8] Anderson R.M., May R.M., Nature (1979) *Journal of Biological Mathematics*, 280, 361-367.
- [9] Anderson R.M., May R.M ., Trans. Roy. Phil. Soc. Series (1981) *Journal of Biological Mathematics*,B 291, 451-524.
- [10] Bailey N.T. (1980) *Journal of Biological Mathematics*, 38, 233-261.
- [11] Cooke K.L. and Yorke (1979) *Journal of Biological Mathematics*, 124-129.
- [12] Hethcote H.W35 (1973) *Journal of Biological Mathematics*, 607-614.
- [13] Blower Sally M. Mclean, Angela, Poroco, Travis C, Peterm Hpoewell, Philip C, sanchez, Melissa A, and Moss, Andrew R.Moss (1995) *Journal of Biological Mathematics*, 35 , 231-254.
- [14] Blower S.M ,Small P.M. and P.C. Hopewell P.C. (1996) *Journal of Biological Mathematics*, 43, 23-45.
- [15] Daley Z, and sally M.Blower (2001) *Journal of Biological Mathematics*, 67, 381-385.
- [16] Anderson R.M. and May R.M. (1986) *Journal of Biological Mathematics*, vol. 134, 533-570.
- [17] Anderson R.M. (1988a) *Journal of Biological Mathematics*, 151, 66-93.
- [18] Anderson R.M. (1988) *Journal of Biological Mathematics*, 1, 241-256.
- [19] Hethcote H.W. (1987) *Journal of Biological Mathematics*, 28, 54-78.
- [20] May R.M. (1988) *Journal of Biological Mathematics*, l331, 665-66.
- [21] Pickering J.,Wiley-J Padion N.S.et al. (1986) *Journal of Biological Mathematics*, 7, 671-698.