

## Mathematical modelling and analysis of the spread of aids caused by co-infection of HIV with TB

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**Abstract-** A combined SIS-SIRS model with migration for the spread of AIDS caused in co- infection involving TB and HIV is proposed in this manuscript. In mathematical modeling, the problem is assumed that an HIV infected individual intract with TB infected person (the former to have HIV & TB). It was assumed further that a fraction of this HIV class becomes AIDS patients. It is also considered that a certain fraction of TB interced person form a reservoir class, a part of which may recover and becomes susceptible again. The model is analysed by using stability theory of differential equation. It was found that AIDS epidemic spread faster due to co- infection of HIV with T.B.

### Introduction

The increasing incidence of various infectious diseases such as gonorrhoea, T.B., HIV, AIDS, etc. increasing major public health problem in most developed and developing countries [1]. The transmission spread of such diseases in human population depends upon various factors such as the number of infectives, susceptible, modes of transmission (social, ecological, and geographical conditions) [1]. An account of modeling and study of epidemic diseases was reported earlier notes by [2] and books by [3]. Many infectious diseases are spread by direct contact between susceptible and infectives, while some others are spread indirectly by carriers, bacteria or vectors [4]. Tuberculosis (TB) is an infectious disease caused by a bacteria called a *Mycobacterium tuberculosis* and affects lung and other parts of the body. In developing countries pulmonary tuberculosis (i.e., T.B.) in the lung accounts more than seventy percent of cases in the world. T.B., which had been controlled to a significant extent earlier, has re-emerged in recent years as one of the leading causes of death. About three million people die every year due to T.B. Any one can become infected with T. B. bacteria but people with H.I.V. infections diseased greater risk of getting infected with this disease. T.B. is transmitted mainly by droplet nuclei when the person suffering from active T.B. breathes, sneezes or coughs in the air [5]. AIDS, the acquired immuno-deficiency syndrome is a fatal disease caused by a retrovirus known as the human immuno deficiency virus (HIV) which although attacks many blood cells it causes havoc on the T-cell in the blood by destroying and decreasing their number leading to decline in body's immunity to fight infection. The term AIDS refers to only the last stage of the HIV infection after which death occurs. The AIDS epidemic is sweeping the world both developed and developing nations as HIV has infected millions of people all over the world without any restriction of nationality, color, caste or religion, etc. HIV infection in adults

(75-80% has been transmitted to one partner through unprotected sexual intercourse when the other partner is infected with HIV. Mother to child transmission (vertical infection) accounts more than 90% of global infection to infants and children. Apart from these, other possible sources of infection are blood transfusion, intravenous drug addiction etc. In general sexually transmitted disease (STD) such as gonorrhoea induce little or no acquired immunity upon recovery but in case of HIV immunity decreases [6]. The transmission dynamics of HIV infection is affected by various factors such as latent period, the number of infected/infectious person in the population, having AIDS, type of sexual activity, type of mixing (homogeneous or heterogeneous), various high risk groups, etc. It is noted that in case of HIV infected persons immunity decreases and they can be early infected by other diseases such as TB forming a class HIV co infected with TB. Some efforts have been taken to understand the transmission of co infection of HIV with TB infections. Mathematical models, taking into account fixed values for the transmission coefficients, these models dealt with the dynamic simulations and the equilibrium point analysis. However, if the equilibrium point analysis could be achieved for all finite transmission coefficients, then it would provide us different scenarios of the interaction between HIV and TB infections. In this manuscript a model for the transmission of co infection of HIV with TB infections and examine some of the epidemiological implications.

### A COMBINED SIS-SIRS MODEL WITH CONSTANT IMMIGRATION

In this section a combined SIS-SIRS model with constant immigration of human population is proposed and analyzed. In proposing the model, the following assumptions are made.

(1). All HIV infected individuals are considered susceptible to TB infection and at the same manner, all TB infected individuals are considered susceptible to HIV.

(2).Due to loss of immunity HIV infected persons get infected with TB after interaction.

(3).There is no separate class of person infected both with HIV and TB. The HIV class also includes persons having co infection.

Based on the above assumptions we propose a mathematical model to assess the spread of co infection HIV with TB infections .It is assumed that the population density N(t) is divided into five classes , the susceptible density X(t),the density of population infected only with TB( $Y_T$ ),the density of population infected with HIV( $Y_H$ ) a fraction of which may be also infected with TB ,the density of AIDS class( $Y_a$ ) and the density of removed class Z(t). The models is proposed as follows,

$$\begin{aligned} \frac{dX}{dt} &= A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + v_1 Y_T + \theta Z \\ \frac{dY_T}{dt} &= \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T - v_2 Y_T \\ \frac{dY_H}{dt} &= \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \\ \frac{dY_a}{dt} &= \mu Y_H - dY_a - \alpha Y_a \\ \frac{dZ}{dt} &= v_2 Y_T - dZ - \theta Z \end{aligned} \quad (1)$$

Where  $N = X + Y_T + Y_H + Y_a + Z$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

in this model ,  $\beta_1$  and  $\beta_2$  are the contact-rates by which susceptible decrease following contact with persons having of TB and AIDS and  $\beta_3$  be the co-infection constant . The clinical AIDS developed at the rate  $\mu$  per year.  $v_1$  is the recovery rate at which TB infectives becomes susceptible again ,  $v_2$  is the rate which infectives goes to removed class the death rate due to , $\theta$  is the recovery rate in removed class ,d is the natural death rate , $\alpha$  be death rate due to AIDS.

### EQUILIBRIUM ANALYSIS

To analyse the model, we consider the following reduced system. Since

$$\begin{aligned} N &= X + Y_T + Y_H + Y_a + Z \\ \frac{dY_T}{dt} &= \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - (d + v_1 + v_2) Y_T \\ \frac{dY_H}{dt} &= \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \\ \frac{dY_a}{dt} &= \mu Y_H - (d + \alpha) Y_a \\ \frac{dZ}{dt} &= v_2 Y_T - dZ - \theta Z \\ \frac{dN}{dt} &= A - dN - \alpha Y_a \end{aligned} \quad (2)$$

The results of the equilibrium analysis are given in the following theorems;

**THEOREM**-There exists following four equilibria of the model discussed before.

$$\begin{aligned} (i) \quad E_0 &= \left( 0, 0, 0, 0, \frac{A}{d} \right) \\ (ii) \quad E_1 &= (Y_T, 0, 0, Z, N) \quad \text{Which exists if } \beta_1 > d + v_1 + v_2 \end{aligned}$$

Where

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1 v}, \quad \bar{Z} = \frac{v_2 Y_T}{d + \theta}, \quad \bar{N} = \frac{A}{d}$$

$$(iii) \quad E_2 = (0, \hat{Y}_H, \hat{Y}_a, 0, N) \quad \text{Which exists if } \beta_2 > d + \mu$$

Where

$$\begin{aligned} \hat{Y}_H &= \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)} \\ \hat{Y}_a &= \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)} \\ \hat{N} &= \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)} \end{aligned}$$

$$(iv) \quad E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*), \quad \text{we consider three cases of this non zero equilibrium as follows;}$$

(a) for  $a_3 = 0$ ,  $E_{30}(Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, Z_0^*, N_0^*)$   
 Where

$$Y_{T0}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}$$

$$Z_0^* = \frac{v_2(a_4 b_1 - a_2 b_2)}{a_1 a_4}, N_0^* = \frac{A}{d} - \frac{\alpha}{d} \frac{\mu b_2}{a_4(\alpha + d)}$$

it exists only when

$$\beta_2 d + \mu \left[ \{v\beta(d+\mu) - \beta_2(d+v_1+v_2)\} \left(1 + \frac{\mu}{\alpha+d}\right) - \beta_3(\beta_2 - d - \mu) \right] > 0$$

(b) for  $0 < E_3(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, Z_1^*, N_1^*)$  Where

$$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_4 + a_2 a_3}, Y_{H1}^* = \frac{a_4 b_2 + a_2 b_3}{a_4 + a_2 a_3}, Z_1^* = \frac{v_2 Y_{T1}^*}{d + \theta}$$

$$N_1^* = \frac{A}{d} - \frac{\alpha \mu Y_{H1}^*}{d(d+\alpha)}, \text{ it exists provided the following conditions are satisfied}$$

$$[v\beta(d+\mu) - \beta_2(d+v_1+v_2)] \left(1 + \frac{\mu}{d+\alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$v\beta\beta \left[ \frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d+\alpha} \right] - \beta_3(\beta_2 - d - \mu) \frac{\alpha\mu}{d} - \mu(d+\alpha)\beta_2 > 0$$

© for  $a_3 < 0$

$E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*)$ , (THE GENERAL CASE)

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$Z_2^* = \frac{v_2 Y_{T2}^*}{d + \theta}, N_2^* = \frac{A}{d} - \frac{\alpha \mu Y_{H2}^*}{d(d+\alpha)}, E_{32} \text{ exists provided that}$$

$a_4 b_1 > a_2 b_2$ , and  $a_4 b_2 > a_3 b_1$ , i.e.

$$[v\beta(d+\mu) - \beta_2(d+v_1+v_2)] \left(1 + \frac{\mu}{d+\alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1 - v_2)}{\beta_1} - (d + \mu)v > 0,$$

where

$$a_1 = v\beta_1, a_2 = \frac{\alpha\mu(\beta_1 - d - v_1 - v_2)}{d(d+\alpha)} + \beta_2 + \frac{\beta_2\mu}{d+\alpha} + \beta_3, a_3 = \beta_2 v - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d+\alpha)} + \beta_2 + \frac{\mu\beta_2}{d+\alpha}, b_1 = \frac{A(\beta_1 - d - v_1 - v_2)}{d}, b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

$$v = 1 + \frac{v_2}{d + \theta}$$

(FOR PROOF SEE APPENDIX—A)

**STABILITY ANALYSIS-**

Now we study the local stability of equilibria

$E_0, E_1, E_2$  and  $E_3$  and the non-linear

stability of non trivial equilibrium  $E_3$ . The results are stated in the form of the following theorems.

**THEOREM - 1;**

The equilibrium  $E_0, E_1, E_2$  is unstable provided the following conditions are satisfied.

(i)  $\beta_1 > d + v_1 + v_2, \beta_2 > d + \mu$

(ii)

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1 - v_2)}{\beta_1} - (d + \mu)v > 0$$

(iii)

$$[v\beta(d+\mu) - \beta_2(d+v_1+v_2)] \left(1 + \frac{\mu}{d+\alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

(FOR PROOF SEE APPENDIX –B)

**THEOREM -2;**

The equilibrium  $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$  is locally asymptotically stable provided that following conditions are satisfied.

(i)

$$[c_1(\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 < \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4N^{*2}}$$

(ii)  $c_1 \beta_1 Y_T^* < \frac{c_3 N^* (d + \alpha)}{4}$

(iii)  $\left( \frac{c_1 \beta_1 Y_T^*}{N^*} - c_4 v_2 \right)^2 < \frac{c_1 c_4 (\theta + d) \beta_1 Y_T^*}{N^*}$

(iv)  $c_1 Y_T^* (\beta_1 - d - v_1 - v_2)^2 < \frac{c_5 \beta_1 N^* d}{4}$

(v)  $\left[ \frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{4 N^*}$

(vi)  $c_2 \beta_2 Y_H^* < \frac{c_4 (d + \theta) N^*}{4}$

(vii)  $c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{c_5 d \beta_2 N^*}{4}$

(viii)  $c_3 c_4 > 0$

(ix)  $c_5 \alpha^2 < \frac{c_3 (d + \alpha)}{4}$

(x)  $c_4 c_5 > 0$

[FOR PROOF SEE APPENDIX –C

**THEOREM;-3**

The equilibrium  $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$  is a non linear stable provided the following conditions are satisfied

(i)  $(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{\beta_1 \beta_2 m_1}{4}$

(ii)  $\left(\frac{\beta_2 m_1}{N^*} - \mu m_2\right)^2 < \frac{\beta_1 m_2 (d + \alpha)}{4 N^*}$

(iii)  $(4\beta_1 + \beta_3)^2 < \frac{d m_3 \beta_1 N^*}{4}$

(iv)  $\left(\frac{-\beta_1}{N^*} + v_2 m_4\right)^2 < \frac{m_4 \beta_1 (d + \theta)}{4 N^*}$

(v)  $\left(\frac{-\beta_2 m_1}{N^*} + \mu m_2\right)^2 < \frac{m_1 m_2 (d + \alpha) \beta_2}{4 N^*}$

(vi)  $(4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} < \frac{d m_3 \beta_2}{4}$

(vii)  $\left(\frac{\beta_2 m_1}{N^*}\right) < \frac{m_4 (d + \theta) \beta_2}{4}$

(viii)  $m_3 \alpha^2 < \frac{m_2 d (d + \alpha)}{4}$

(ix)  $m_2 m_4, m_3 m_4 > 0$

to prove the above theorem we need the following leema,

the region of attraction is given by the following set.

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}, 0 \leq Y_H \leq \frac{A(\beta_2 - d - \mu)}{d\beta_2}, 0 \leq Y_a \leq \frac{\mu A(\beta_2 - d - \mu)}{d\beta_2(d + \alpha)}$$

$$0 \leq Z \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1(d + \theta)}, 0 \leq N \leq \frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

**A PARTICULAR CASE**

**SIS MODEL WITH CONSTANT IMMIGRATION;**

If in the model if we put  $v_2 = 0$ ,  $\theta=0$  it becomes sis model .

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H) X}{N} - dX + v_1 Y_T$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$$

where

$$N = X + Y_T + Y_H + Y_a$$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a$$

all the symbols have their same meaning as in the previous model.

**EQUILIBRIUM ANALYSIS;**

In this case there exists four equilibria of the model.

1;  $E_0 = \left(0, 0, 0, \frac{A}{d}\right)$

2;  $E_1 = (Y_T, 0, 0, N)$  which exists if  $\beta_1 > d + v_1$

where

$$Y_T = \frac{A(\beta_1 - d - v_1)}{d\beta_1}, N = \frac{A}{d}$$

3;  $E_2 = (0, \hat{Y}_H, \hat{Y}_a, N)$ , Which exists if  $\beta_2 > d + \mu$

where

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

4;  $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$ , This is defined in three parts ;

(a)  $E_{30} (Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, N_0^*)$  Where

$$Y_{T0}^* = \frac{b a_4 - a b_2}{a a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}, N_0^* = \frac{A}{d} \frac{\alpha}{d a_4(\alpha + d)}$$

it exists only when

$$\beta_2 \rangle d + \mu$$

$$\left[ \beta_1(d + \mu) - \beta_2(d + \nu_1) \right] \left( 1 + \frac{\mu}{\alpha + d} \right) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

(b)

$E_3(Y_T^*, Y_H^*, Y_a^*, N_1^*)$  Where

$$Y_T^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}, Y_H^* = \frac{a_1 b_2 + a_3 b_1}{a_1 a_4 + a_2 a_3}, Y_a^* = \frac{\mu Y_H^*}{d + \alpha}$$

$$N_1^* = \frac{A}{d} \frac{\alpha \mu Y_H^*}{d(d + \alpha)}, \text{ it exist provided that following conditions satisfy}$$

$$[\beta_1(d + \mu) - \beta_2(d + \nu_1)] \left( 1 + \frac{\mu}{d + \alpha} \right) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

$$\beta_2 \beta_3 \left[ \frac{\alpha \mu}{d} + \alpha + d + \frac{\mu}{d + \alpha} \right] - \beta_3(\beta_2 - d - \mu) \frac{\alpha \mu}{d} - \mu(d + \alpha) \beta_2 \rangle 0$$

©

$E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*)$ , (THE GENERAL CASE)

Where

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$N_2^* = \frac{A}{d} \frac{\alpha \mu Y_{H2}^*}{d(d + \alpha)}, \text{ it exist provided that}$$

$a_4 b_1 \rangle a_2 b_2$ , and  $a_1 b_2 \rangle a_3 b_1$ , i.e.

$$[\beta_1(d + \mu) - \beta_2(d + \nu_1)] \left( 1 + \frac{\mu}{d + \alpha} \right) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - \nu_1 - \nu_2)}{\beta_1} - (d + \mu) \rangle 0,$$

where

$$a_1 = \beta_1, a_2 = \frac{\alpha \mu (\beta_1 - d - \nu_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1 \mu}{d + \alpha} + \beta_3, a_3 = \beta_2 - \beta_3$$

$$a_4 = \frac{\alpha \mu (\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu \beta_2}{d + \alpha}, b_1 = \frac{A(\beta_1 - d - \nu_1)}{d}, b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

(FOR PROOF SEE APPENDIX—A)

**STABILITY ANALYSIS;-**

now we study the local stability of equilibria

$E_0, E_1, E_2$  and  $E_3$  and the non-linear

stability of non trivial equilibrium  $E_3$ . The results are stated in the form of the following theorems .

**THEOREM - 1;**

The equilibrium  $E_0, E_1, E_2$  is unstable provided the following conditions are satisfied.

(i)  $\beta_1 \rangle d + \nu_1, \beta_2 \rangle d + \mu$

(ii)  $\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - \nu_1)}{\beta_1} - (d + \mu) \rangle 0$

(iii)

$$[\beta_1(d + \mu) - \beta_2(d + \nu_1)] \left( 1 + \frac{\mu}{d + \alpha} \right) - \beta_3(\beta_2 - d - \mu) \rangle 0$$

(FOR PROOF SEE APPENDIX –B)

**THEOREM -2;**

The equilibrium  $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$  is locally asymptotically stable provided that following conditions are satisfied.

(i)

$$[c_1(\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2(\beta_2 - \beta_3) \frac{Y_H^*}{N^*}]^2 \langle \frac{4c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{9N^{*2}}$$

(ii)  $c_1 \beta_1 Y_T^* \langle \frac{4c_3 N^* (d + \alpha)}{9}$

(iii)  $c_1 Y_T^* (\beta_1 - d - \nu_1)^2 \langle \frac{4c_4 \beta_1 N^* d}{9}$

(iv)

$$[\frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3]^2 \langle \frac{4c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{9N^{*2}}$$

(v)  $c_2 Y_H^* (\beta_2 - d - \mu)^2 \langle \frac{4c_4 d \beta_2 N^*}{9}$

(vi)  $c_4 \alpha^2 \langle \frac{4c_3 (d + \alpha)}{9}$

(FOR PROOF SEE APPENDIX-C)

**THEOREM-3**

The equilibrium  $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$  is a non linearly stable provided the following conditions are satisfied

(i)  $(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 \langle \frac{4\beta_1 \beta_2 m_1}{9}$

(ii)  $\beta_1 \langle \frac{4m_2 (d + \alpha) N^*}{9}$

(iii)  $(3\beta_1 + \beta_3)^2 \langle \frac{4dm_3 \beta_1 N^*}{9}$

(iv)

$$\left( \frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 \langle \frac{4m_1 m_2 (d + \alpha) \beta_2}{9N^{*2}}$$

(v)

$$(\beta_2 m_1 + \beta_2 m_2 + \beta_3 m_1)^2 \langle \frac{4\beta_2 m_1 m_3 d N^*}{9}$$

(vi)  $m_3 \alpha^2 \langle \frac{4m_2 d (d + \alpha)}{9}$

To prove the above theorem we need the following lemma.

The region of attraction is

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1)}{d\beta_1}, 0 \leq Y_H \leq \frac{A(\beta_2 - d - \mu)}{d\beta_2}, 0 \leq Y_a \leq \frac{\mu A(\beta_2 - d - \mu)}{d\beta_2(d + \alpha)}, 0 \leq N \leq \frac{A}{d}$$

[FOR PROOF SEE APPENDIX-D]

**CONCLUSION**

In this paper, a non linear model is proposed and analyzed to study the co- infection of TB and AIDS. It is shown that the spread of AIDS increases due to co-infection of TB and AIDS and the disease becomes endemic. It is also found that due to migration of HIV infection the spread of AIDS increases further.

**APPENDIX-A**

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H)X}{N} - dX + v_1 Y_T + \theta Z \quad (A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T - v_2 Y_T \quad (A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad (A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad (A_4)$$

$$\frac{dZ}{dt} = v_2 Y_T - dZ - \theta Z \quad (A_5)$$

Where  $N = X + Y_T + Y_H + Y_a + Z$

so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A_6)$$

the reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1 + v_2) Y_T \quad (A_7)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \quad (A_8)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha) Y_a \quad (A_9)$$

$$\frac{dZ}{dt} = v_2 Y_T - dZ - \theta Z \quad (A_{10})$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A_{11})$$

$$E_0(0,0,0,0, \frac{A}{d})$$

**EXISTENCE OF**

It is obvious from the model  $(A_7 \dots \dots A_{11})$ ,

we can find if  $Y_T = Y_H = Y_a = 0$

So  $N = \frac{A}{d}$

This is disease free equilibrium.

**EXISTENCE OF  $E_1(\bar{Y}_T, 0, 0, \bar{Z}, \bar{N})$**

Since  $Y_H = 0 \Rightarrow Y_a = 0$  and  $Y_T \neq 0$   
From

$(A_{11})$  we get  $\bar{N} = \frac{A}{d}$ , From

$(A_{10})$ , we get  $Z = \frac{v_2 Y_T}{\theta + d}$

From  $(A_7)$  we get

$$\beta_1 Y_T (\bar{N} - \bar{Y}_T - \bar{Z}) - (d + v_1 + v_2) \bar{Y}_T \bar{N} = 0$$

Since  $\bar{Y}_T \neq 0$

So on solving, we get

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1 v}, \bar{Z} = \frac{v_2 Y_T}{d + \theta}, \bar{N} = \frac{A}{d}$$

Which exists if  $\beta_1 > d + v_1 + v_2$

**EXISTENCE OF  $E_2(0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$**

Since

$$\hat{Y}_T = 0 \Rightarrow \hat{Z} = 0, \text{ And } \hat{Y}_H, \hat{Y}_a \neq 0$$

So, on solving  $A_8, A_9, A_{11}$  We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha \mu (\beta_2 - d - \mu) + \beta_2 d (\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A \mu (\beta_2 - d - \mu)}{\alpha \mu (\beta_2 - d - \mu) + \beta_2 d (\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d (\alpha + d + \mu)}{\alpha \mu (\beta_2 - d - \mu) + \beta_2 d (\alpha + d + \mu)}$$

, Which exists if  $\beta_2 > d + \mu$

**EXISTENCE OF  $E_3 = (Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$**

From  $(A_9)$ , we get  $Y_a^* = \frac{\mu Y_H^*}{d + \alpha}$ , from  $(A_{10})$  we get  $Z^* = \frac{v_2 Y_T^*}{d + \theta}$

From  $(A_{11})$   $N^* = \frac{A}{d} - \frac{\alpha Y_H^*}{d + \alpha}$

So  $A_7$  and  $A_8$  becomes

$$\beta_1 v Y_T^* + Y_H^* \left[ \frac{\alpha \mu (\beta_1 - d - v_1 - v_2)}{d(d + \alpha)} + \beta_1 + \frac{\beta_2 \mu}{d + \alpha} + \beta_3 \right] = \frac{A(\beta_1 - d - v_1 - v_2)}{d} \quad (A_{12})$$

$$Y_T^* (\beta_2 v - \beta_3) + Y_H^* \left[ \frac{\alpha A (\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu \beta_3}{d + \alpha} \right] = \frac{A(\beta_2 - d - \mu)}{d} \quad (A_{13})$$

where

$$a_1 = v\beta, a_2 = \frac{\alpha\mu(\beta-d-v_1-v_2)}{d(d+\alpha)} + \beta + \frac{\beta\mu}{d+\alpha}, a_3 = \beta_2 v - \beta_3$$

$$a_4 = \frac{\alpha\mu(\beta_2-d-\mu)}{d(d+\alpha)} + \beta_2 + \frac{\mu\beta_2}{d+\alpha}, b_1 = \frac{A(\beta-d-v_1-v_2)}{d}, b_2 = \frac{A(\beta_2-d-\mu)}{d}$$

$$v = 1 + \frac{v_2}{d+\theta}$$

so  $A_{12}$  And  $A_{13}$  Becomes

$$a_1 Y_T^* + a_2 Y_H^* = b_1, a_3 Y_T^* + a_4 Y_H^* = b_2 \dots \dots \dots (A_4)$$

now here arise the three cases –  
(1); if

$$a_3 = 0 \Rightarrow \beta_2 v = \beta_3 \text{ so the equilibrium } E_3(Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, Z_0^*, N_0^*)$$

so

$$Y_{T0}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha+d)}$$

$$Z_0^* = \frac{v_2(b_1 a_4 - a_2 b_2)}{a_1 a_4}, N_0^* = \frac{A}{d} - \frac{\alpha}{d} - \frac{\mu b_2}{a_4(\alpha+d)}$$

it exists only when

$$\beta_2 d + \mu \left[ \{v\beta(d+\mu) - \beta_2(d+v_1+v_2)\} \left(1 + \frac{\mu}{\alpha+d}\right) - \beta_3(\beta_2-d-\mu) \right] > 0$$

(2) if  $\alpha < 0$  so the equilibrium  $E_5(Y_{T1}^*, Y_{H1}^*, Y_{a1}^*, Z_1^*, N_1^*)$  Where

$$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}, Y_{H1}^* = \frac{a_2 b_2 + a_3 b_1}{a_1 a_4 + a_2 a_3}, Z_1^* = \frac{v_2 Y_{T1}^*}{d+\theta}$$

$$N_1^* = \frac{A}{d} - \frac{\alpha Y_{T1}^*}{d(d+\alpha)}, \text{ it exists provided that following condition is satisfied}$$

$$[v\beta(d+\mu) - \beta_2(d+v_1+v_2)] \left(1 + \frac{\mu}{d+\alpha}\right) - \beta_3(\beta_2-d-\mu) > 0$$

$$v\beta \left[ \frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d+\alpha} \right] - \beta_3(\beta_2-d-\mu) - \frac{\alpha\mu}{d} - \mu(d+\alpha)\beta_2 > 0$$

(3) if  $a_3 > 0$  so the equilibrium  $E_{32}(Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, Z_2^*, N_2^*)$ , (THE GENERAL CASE)

Where

$$Z_2^* = \frac{v_2 Y_{T2}^*}{d+\theta}, N_2^* = \frac{A}{d} - \frac{\alpha\mu Y_{H2}^*}{d(d+\alpha)}, \text{ it exists provided that}$$

$a_4 b_1 > a_2 b_2$ , and  $a_1 b_2 > a_3 b_1$ , i.e.

$$[v\beta(d+\mu) - \beta_2(d+v_1+v_2)] \left(1 + \frac{\mu}{d+\alpha}\right) - \beta_3(\beta_2-d-\mu) > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1-d-v_1-v_2)}{\beta_1} - (d+\mu)v > 0,$$

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d+\alpha}$$

**APPENDIX-B**

Using the model  $(A_7, \dots, A_{11})$ , The general jacobian matrix can be written as follows;

$$M = \begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ 0 & \mu & -(\alpha+d) & 0 & 0 \\ v_2 & 0 & 0 & -(d+\theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

where

$$m_{11} = \frac{\beta(N-Y_T-Y_H-Y_a-Z)}{N} - \frac{\beta Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - v_1 - v_2$$

$$m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}, m_{14} = \frac{-\beta_1 Y_T}{N}$$

$$m_{15} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T(N-Y_T-Y_H-Y_a-Z)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2(N-Y_T-Y_H-Y_a-Z)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d+v)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}, m_{24} = \frac{-\beta_2 Y_H}{N}$$

$$m_{25} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H(N-Y_T-Y_H-Y_a-Z)}{N^2} -$$

$$\frac{\beta_3 Y_T Y_H}{N^2}$$

**PROOF OF THEOREM-1 stability of**

$$E_0(0,0,0,0, \frac{A}{d})$$

The jacobian matrix for  $E_0$  is given by

$$M(E_0) = \begin{matrix} \beta-d-v_1-v_2 & 0 & 0 & 0 & 0 \\ 0 & \beta_2-d-\mu & 0 & 0 & 0 \\ 0 & \mu & -(d+\alpha) & 0 & 0 \\ v_2 & 0 & 0 & -(d+\theta) & 0 \\ 0 & 0 & -\alpha & 0 & -d \end{matrix}$$

The eigen values are

$$\beta_1 - d - v_1 - v_2, \beta_2 - d - \mu, -(d+\alpha), -(d+\theta), -d$$

since  $\beta_1 > d + v_1 + v_2$ ,  $\beta_2 > d + \mu$  so the equilibrium  $E_0$  is unstable.

**Stability for  $E_1(\bar{Y}_T, 0, 0, \bar{Z}, \bar{N})$**

The jacobian matrix for  $E_1$  is given as

$$M(E_1) = \begin{matrix} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ \begin{matrix} 0 \\ \mu \\ v_2 \\ 0 \end{matrix} & \begin{matrix} b_{22} \\ \mu \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ -(\alpha+d) \\ 0 \\ -\alpha \end{matrix} & \begin{matrix} 0 \\ 0 \\ -(d+\theta) \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ -d \end{matrix} \end{matrix}$$

Where

$$b_{11} = -(\beta_1 - d - v_1 - v_2), b_{12} = -v\beta + \beta_1(\beta_1 - d - v_1 - v_2), b_{13} = b_{14} = -(\beta_1 - d - v_1 - v_2)$$

$$b_{15} = \frac{(\beta_1 - d - v_1 - v_2)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1} d + \frac{\beta_3}{\beta_1}$$

$$(\beta_1 - d - v_1 - v_2) - v(d + \mu)$$

Since eigen value are

$$b_{11}, b_{22}, -(\alpha + d), -(\theta + d), -d$$

Since  $b_{22} > 0$  so  $E_1$  is unstable

**Stability of  $E_2(0, \hat{Y}_H, \hat{Y}_a, 0, \hat{N})$**

The jacobian for  $E_2$  is

$$M(E_2) = \begin{matrix} & \hat{a}_1 & 0 & 0 & 0 & 0 \\ \begin{matrix} \hat{b}_1 \\ \mu \\ v_2 \\ 0 \end{matrix} & \begin{matrix} \hat{b}_2 \\ \mu \\ 0 \\ 0 \end{matrix} & \begin{matrix} \hat{b}_3 \\ -(\alpha+d) \\ 0 \\ -\alpha \end{matrix} & \begin{matrix} \hat{b}_4 \\ 0 \\ -(\theta+d) \\ 0 \end{matrix} & \begin{matrix} \hat{b}_5 \\ 0 \\ 0 \\ -d \end{matrix} \end{matrix}$$

Where

$$\hat{a}_1 = (\mu + d)v\beta - \frac{\beta_3(\beta_1 + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta_2(d + v_1 + v_2), \hat{b}_1 = \frac{\beta_3 - \beta_2}{\beta_2} \frac{(\alpha + d)(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu},$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}, \hat{b}_4 =$$

$$\frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is  $a_1$  which is  $>0$  i.e.  $E_2$  is unstable

### APPENDIX-C

**Local stability of Equilibrium  $E_3(Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$  By Lyapunov method,**

The model  $(A_7 - A_{11})$  can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, Y_H = Y_H^* + y_H, Y_a = Y_a^* + y_a, Z = Z^* + z, N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}Z + m_{15}n$$

$$\frac{dY_H}{dt} = m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}Z + m_{25}n$$

$$\frac{dy_a}{dt} = \mu y_H - (d + \alpha)y_a$$

$$\frac{dz}{dt} = v_2 y_T - (d + \theta)z$$

$$\frac{dn}{dt} = -\alpha Y_a - dn$$

Where we use following Lyapunov function ;

$$V = \frac{c_1 y_T^2}{2} + \frac{c_2 y_H^2}{2} + \frac{c_3 y_a^2}{2} + \frac{c_4 z^2}{2} + \frac{c_5 n^2}{2}$$

Where  $c_1, c_2, c_3, c_4, c_5$  are to be determined. So

$$\dot{V} = c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 z \frac{dz}{dt} + c_5 n \frac{dn}{dt}$$

$$= c_1 y_T (m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}z + m_{15}n) + c_2 y_H (m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}z + m_{25}n)$$

+



$$\begin{aligned}
 & c_3 y_a (\mu y_H - (d + \alpha) y_a) + c_4 z (v_2 y_T - (\theta + d) z) + c_5 n (-\alpha y_a - dn) \\
 &= \left[ \frac{1}{4} c m_1 y_T^2 + y_H (c m_2 + c m_3) + \frac{1}{4} c m_2 y_H^2 \right] + \left[ \frac{1}{4} c m_1 y_T^2 + y_H (c m_3) - \frac{1}{4} c_3 (d + \alpha) y_a^2 \right] \\
 &+ \left[ \frac{1}{4} c m_1 y_T^2 + y_H (c m_4 + c_4 v_2) - \frac{1}{4} c_3 (d + \theta) z^2 \right] + \left[ \frac{1}{4} c m_1 y_T^2 + y_H (c m_3) - \frac{1}{4} c_4 d n^2 \right] \\
 &+ \left[ \frac{1}{4} c m_2 y_H^2 + y_H (c m_3 + \mu_3) - \frac{1}{4} c_3 (d + \alpha) y_a^2 \right] + \left[ \frac{1}{4} c m_2 y_H^2 + y_H (c m_3) - \frac{1}{4} c_4 (d + \theta) z^2 \right] \\
 &+ \left[ \frac{1}{4} c m_2 y_H^2 + y_H (c m_3) - \frac{1}{4} c_4 d n^2 \right] + \left[ \frac{1}{4} c_3 (d + \alpha) y_a^2 - \frac{1}{4} c_4 (d + \theta) z^2 \right] \\
 &+ \left[ \frac{1}{4} c_3 (d + \alpha) y_a^2 + y_H (c_4 \theta - \frac{1}{4} c_4 d n^2) \right] + \left[ \frac{1}{4} c_4 (d + \theta) z^2 + \frac{1}{4} c_4 d n^2 \right]
 \end{aligned}$$

for the equilibrium  $(Y_T^*, Y_H^*, Y_a^*, Z^*, N^*)$

$$m_1 = \frac{-\beta Y_T^*}{N^*}, m_2 = (-\beta - \beta_3) \frac{Y_T^*}{N^*}, m_3 = \frac{-\beta Y_T^*}{N^*}, m_4 = m_5 = (\beta - d - v_1 - v_2) \frac{Y_T^*}{N^*}$$

$$m_6 = (-\beta_2 + \beta) \frac{Y_H^*}{N^*}, m_7 = m_8 = m_9 = \frac{\beta Y_H^*}{N^*}, m_{10} = (\beta_2 - d - \mu) \frac{Y_H^*}{N^*}$$

since all the coefficients are negative and will be negative if the provided conditions are satisfied

(i)

$$\left[ c_1 (\beta_1 + \beta_3) \frac{Y_T^*}{N^*} + c_2 (\beta_2 - \beta_3) \frac{Y_H^*}{N^*} \right]^2 < \frac{c_1 c_2 \beta_1 \beta_2 Y_T^* Y_H^*}{4 N^{*2}}$$

$$(ii) \quad c_1 \beta_1 Y_T^* < \frac{c_3 N^* (d + \alpha)}{4}$$

$$(iii) \quad \left( \frac{c_1 \beta_1 Y_T^*}{N^*} - c_4 v_2 \right)^2 < \frac{c_1 c_4 (\theta + d) \beta_1 Y_T^*}{N^*}$$

$$(iv) \quad c_1 Y_T^* (\beta_1 - d - v_1 - v_2)^2 < \frac{c_5 \beta_1 N^* d}{4}$$

$$(v) \quad \left[ \frac{c_2 \beta_2 Y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{c_2 c_3 (d + \alpha) \beta_2 Y_H^*}{4 N^*}$$

$$(vi) \quad c_2 \beta_2 Y_H^* < \frac{c_4 (d + \theta) N^*}{4}$$

$$(vii) \quad c_2 Y_H^* (\beta_2 - d - \mu)^2 < \frac{c_5 d \beta_2 N^*}{4}$$

$$(viii) \quad c_3 c_4 > 0$$

$$(ix) \quad c_5 \alpha^2 < \frac{c_3 (d + \alpha)}{4} \quad (x) \quad c_4 c_5 > 0$$

**APPENDIX-D  
(NON LINEAR STABILITY)  
PROOF OF THE LEEMA**

From model  $(A_7 \dots A_9)$  we can obtain as follows;

$$\frac{dN}{dt} = A - dN - \alpha Y_a \leq A - dN \Rightarrow N \leq \frac{A}{d}$$

$$\frac{dY_T}{dt} = \frac{\beta Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1 + v_2) Y_T$$

$$\leq (\beta_1 - d - v_1 - v_2) Y_T - \frac{\beta_1 Y_T^2}{N}$$

$$\Rightarrow Y_T \leq \frac{A (\beta_1 - d - v_1 - v_2)}{d \beta_1}$$

$$Y_H \leq \frac{(\beta_2 - d - \mu) A}{\beta_2 d}$$

Similarly and

$$\frac{dY_a}{dt} = \mu Y_H - d Y_a - \alpha Y_a$$

$$\Rightarrow Y_a \leq \frac{(\beta_2 - d - \mu) A \mu}{\beta_2 d (d + \alpha)}$$

$$\frac{dZ}{dt} = v_2 Y_T - d Z - \theta Z \Rightarrow Z \leq$$

$$\frac{A (\beta_1 - d - v_1 - v_2) v_2}{d \beta_1 (d + \theta)}$$

so the region of attraction is

$$0 \leq Y_T \leq \frac{A (\beta_1 - d - v_1 - v_2)}{d \beta_1}$$

$$0 \leq Y_H \leq \frac{(\beta_2 - d - \mu) A}{\beta_2 d}$$

$$0 \leq Y_a \leq \frac{(\beta_2 - d - \mu) A \mu}{\beta_2 d (d + \alpha)}, \quad 0 \leq Z \leq$$

$$\frac{A (\beta_1 - d - v_1 - v_2) v_2}{d \beta_1 (d + \theta)}$$

$$0 \leq N \leq \frac{A}{d}$$

**PROOF OF THE THEOREM**

In this case we use the following Lyapunov function;

$$W = (Y_T - Y_T^* - Y_T^* \log \frac{Y_T}{Y_T^*}) + m_1 (Y_H - Y_H^* - Y_H^* \log \frac{Y_H}{Y_H^*}) + \frac{m_2}{2} (Y_a - Y_a^*)^2 + \frac{m_3}{2} (N - N^*)^2 + \frac{m_4}{2} (Z - Z^*)^2$$

on differentiating and substituting the values from model we get

$$\begin{aligned}
 & \left[ \frac{-\beta}{4N} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_H - Y_H^*) \left( \frac{-\beta - \beta_3 - \beta_2 m_1 + \beta m_1}{N} + \frac{1}{4} (Y_T - Y_T^*) \left( \frac{-\beta m_1}{N} \right) \right) \right] \\
 & + \left[ \frac{-\beta}{4N} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_a - Y_a^*) \left( \frac{-\beta m_1}{N} + \mu m_2 \right) - \frac{1}{4} m_3 (d + \alpha) (Y_a - Y_a^*)^2 \right] \\
 & + \left[ \frac{-\beta}{4N} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (N - N^*) \left( \frac{\beta Y_T + \beta_3 Z + \beta_2 Y_a + (\beta + \beta_3) Y_H}{NN} + \frac{1}{4} d m_1 (N - N^*)^2 \right) \right] \\
 & + \left[ \frac{-\beta}{4N} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Z - Z^*) \left( \frac{-\beta}{N} + v_2 m_4 \right) - \frac{1}{4} m_4 (d + \theta) (Z - Z^*)^2 \right] \\
 & + \left[ \frac{1}{4} (Y_H - Y_H^*)^2 \left( \frac{-\beta m_1}{N} \right) + (Y_H - Y_H^*) (Y_a - Y_a^*) \left( \frac{-\beta m_1}{N} + \mu m_2 \right) - \frac{1}{4} m_3 (d + \alpha) (Y_a - Y_a^*)^2 \right] \\
 & + \left[ \frac{1}{4} (Y_H - Y_H^*)^2 \left( \frac{-\beta m_1}{N} \right) + (Y_H - Y_H^*) (N - N^*) \left( \frac{\beta m_1 Y_T + (Y_H + Y_a + Z) \beta m_1 - \beta m_1 Y_H}{NN} + \frac{1}{4} d m_1 (N - N^*)^2 \right) \right] \\
 & + \left[ \frac{1}{4} (Y_H - Y_H^*)^2 \left( \frac{-\beta m_1}{N} \right) + (Y_H - Y_H^*) (Z - Z^*) \left( \frac{-\beta m_1}{N} \right) - \frac{1}{4} m_4 (d + \theta) (Z - Z^*)^2 \right] \\
 & + \left[ -\frac{1}{4} m_2 (d + \alpha) (Y_a - Y_a^*)^2 + (Y_a - Y_a^*) (N - N^*) \left( -m_3 \alpha - \frac{1}{4} d m_1 (N - N^*)^2 \right) \right] \\
 & + \left[ \frac{1}{4} m_4 (d + \alpha) (Y_a - Y_a^*)^2 + \frac{1}{4} m_4 (d + \theta) (Z - Z^*)^2 + \left[ \frac{1}{4} d m_1 (N - N^*)^2 + \frac{1}{4} d m_1 (N - N^*)^2 \right] \right]
 \end{aligned}$$

$$\frac{Y_T}{N}, \frac{Y_H}{N}, \frac{Y_a}{N}, \frac{Z}{N} = 1$$

CONSIDER

$W$  is negative definite if the following conditions are satisfied ;

(i)  $(\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{\beta_1 \beta_2 m_1}{4}$

(ii)  $\left(\frac{\beta_2 m_1}{N^*} - \mu m_2\right)^2 < \frac{\beta_1 m_2 (d + \alpha)}{4 N^*}$

(iii)  $(4\beta_1 + \beta_3)^2 < \frac{d m_3 \beta_1 N^*}{4}$

(iv)  $\left(\frac{-\beta_1}{N^*} + v_2 m_4\right)^2 < \frac{m_4 \beta_1 (d + \theta)}{4 N^*}$

(v)  $\left(\frac{-\beta_2 m_1}{N^*} + \mu m_2\right)^2 < \frac{m_1 m_2 (d + \alpha) \beta_2}{4 N^*}$

(vi)  $(4\beta_2 - \beta_3)^2 \frac{m_1}{N^*} < \frac{d m_3 \beta_2}{4}$

(vii)  $\left(\frac{\beta_2^2 m_1}{N^*}\right) < \frac{m_4 (d + \theta) \beta_2}{4}$

(viii)  $m_3 \alpha^2 < \frac{m_2 d (d + \alpha)}{4}$

(ix)  $m_2 m_4, m_3 m_4 > 0$

**FOR SIS MODEL;**

**APPENDIX-A1**

The given model is

$$\frac{dX}{dt} = A - \frac{(\beta_1 Y_T + \beta_2 Y_H) X}{N} - dX + v_1 Y_T \quad (A_1)$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T X}{N} - \frac{\beta_3 Y_T Y_H}{N} - dY_T - v_1 Y_T \quad (A_2)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H X}{N} + \frac{\beta_3 Y_T Y_H}{N} - dY_H - \mu Y_H \quad (A_3)$$

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a \quad (A_4)$$

Where  $N = X + Y_T + Y_H + Y_a$   
so

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A_5)$$

the reduced system is given below,

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1) Y_T \quad (A)$$

$$\frac{dY_H}{dt} = \frac{\beta_2 Y_H (N - Y_T - Y_a - Z)}{N} + \frac{\beta_3 Y_T Y_H}{N} - (d + \mu) Y_H \quad (A)$$

$$\frac{dY_a}{dt} = \mu Y_H - (d + \alpha) Y_a \quad (A)$$

$$\frac{dN}{dt} = A - dN - \alpha Y_a \quad (A)$$

$$E_0(0,0,0, \frac{A}{d})$$

**EXISTENCE OF**

It is obvious from the model  $(A_6 \dots \dots \dots A_9)$ ,

we can find if  $Y_T = Y_H = Y_a = 0$

So  $N = \frac{A}{d}$

This is disease free equilibrium.

**EXISTENCE OF  $E_1(\bar{Y}_T, 0, 0, \bar{N})$**

Since  $Y_H = 0 \Rightarrow Y_a = 0$  and  $Y_T \neq 0$

$(A_9)$  we get  $\bar{N} = \frac{A}{d}$ ,

From

from  $(A_6)$  we get

$$\beta_1 Y_T (\bar{N} - \bar{Y}_T) - (d + v_1) \bar{Y}_T \bar{N} = 0$$

Since  $\bar{Y}_T \neq 0$

So on solving , we get

$$\bar{Y}_T = \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1 v}, \bar{N} = \frac{A}{d}$$

Which exists if  $\beta_1 > d + v_1$

**EXISTENCE OF  $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$**

Since  $\hat{Y}_T = 0$  And  $\hat{Y}_H, \hat{Y}_a \neq 0$

So ,on solving  $A_7, A_8, A_9$  We get,

$$\hat{Y}_H = \frac{A(\alpha + d)(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{Y}_a = \frac{A\mu(\beta_2 - d - \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

$$\hat{N} = \frac{\beta_2 d(\alpha + d + \mu)}{\alpha\mu(\beta_2 - d - \mu) + \beta_2 d(\alpha + d + \mu)}$$

Which exists if  $\beta_2 > d + \mu$

**EXISTENCE OF**  $E_3 = (Y_T^*, Y_H^*, Y_a^*, N^*)$

(A<sub>8</sub>), we get  $Y_a^* = \frac{\mu Y_H^*}{d + \alpha}$ ,

From (A<sub>9</sub>)  $N^* = \frac{A}{d} - \frac{\alpha\mu Y_H^*}{d(d + \alpha)}$

From

So  $A_6$  and  $A_7$  Becomes

$$\beta_1 Y_T^* + Y_H^* \left[ \frac{\alpha\mu(\beta_1 - d - v_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1\mu}{d + \alpha} + \beta_3 \right] = \frac{A(\beta_1 - d - v_1)}{d} \dots \dots \dots (A_{10})$$

$$Y_T^*(\beta_2 - \beta_1) + Y_H^* \left[ \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha} \right] = \frac{A(\beta_2 - d - \mu)}{d} \dots \dots \dots (A_{11})$$

where

$$a_1 = \beta_1, a_2 = \frac{\alpha\mu(\beta_1 - d - v_1)}{d(d + \alpha)} + \beta_1 + \frac{\beta_1\mu}{d + \alpha} + \beta_3, a_3 = \beta_2 - \beta_1$$

$$a_4 = \frac{\alpha\mu(\beta_2 - d - \mu)}{d(d + \alpha)} + \beta_2 + \frac{\mu\beta_2}{d + \alpha}, b_1 = \frac{A(\beta_1 - d - v_1)}{d}, b_2 = \frac{A(\beta_2 - d - \mu)}{d}$$

so  $A_{12}$  And  $A_{13}$  Becomes

$$a_1 Y_T^* + a_2 Y_H^* = b_1, a_3 Y_T^* + a_4 Y_H^* = b_2 \dots \dots \dots (A_{12})$$

now here arise the three cases –

(1);if

$$a_3 = 0 \Rightarrow \beta_2 = \beta_1, \text{ so the equilibrium } E_{30} (Y_{T0}^*, Y_{H0}^*, Y_{a0}^*, N_0^*)$$

so

$$Y_{T0}^* = \frac{b_1 a_4 - a_2 b_2}{a_1 a_4}, Y_{H0}^* = \frac{b_2}{a_4}, Y_{a0}^* = \frac{\mu b_2}{a_4(\alpha + d)}$$

$$Z_0^* = \frac{v_2(b_1 a_4 - a_2 b_2)}{a_1 a_4}, N_0^* = \frac{A}{d} - \frac{\alpha}{d} \frac{\mu b_2}{a_4(\alpha + d)}$$

it exists only when

$$\beta_2 > d + \mu$$

$$\left[ \{\beta_1(d + \mu) - \beta_2(d + v_1)\} \left(1 + \frac{\mu}{\alpha + d}\right) - \beta_3(\beta_2 - d - \mu) > 0 \right]$$

(2) if  $a_3 < 0$  so The Equilibrium  $E_3 (Y_T^*, Y_H^*, Y_a^*, N_1^*)$ , Where

$$Y_{T1}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 + a_2 a_3}, Y_{H1}^* = \frac{a_1 b_2 + a_3 b_1}{a_1 a_4 + a_2 a_3},$$

$$N_1^* = \frac{A}{d} - \frac{\alpha\mu Y_{H1}^*}{d(d + \alpha)}, \text{ it exists provided the following condition is satisfied}$$

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{d + \alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\beta_1 \beta_2 \left[ \frac{\alpha\mu}{d} + \alpha + d + \frac{\mu}{d + \alpha} \right] - \beta_3(\beta_2 - d - \mu) \frac{\alpha\mu}{d} - \mu(d + \alpha)\beta_2 > 0$$

(3) if  $a_3 > 0$  so The Equilibrium  $E_{32} (Y_{T2}^*, Y_{H2}^*, Y_{a2}^*, N_2^*)$ , (THE GENERAL CASE)

Where

$$Y_{T2}^* = \frac{a_4 b_1 - a_2 b_2}{a_1 a_4 - a_2 a_3}, Y_{H2}^* = \frac{a_1 b_2 - a_3 b_1}{a_1 a_4 - a_2 a_3}, Y_{a2}^* = \frac{\mu Y_{H2}^*}{d + \alpha}$$

$$N_2^* = \frac{A}{d} - \frac{\alpha\mu Y_{H2}^*}{d(d + \alpha)}, \text{ it exists provided that}$$

$a_4 b_1 > a_2 b_2$ , and  $a_1 b_2 > a_3 b_1$ , i.e.

$$[\beta_1(d + \mu) - \beta_2(d + v_1)] \left(1 + \frac{\mu}{d + \alpha}\right) - \beta_3(\beta_2 - d - \mu) > 0$$

$$\frac{\beta_2 d}{\beta_1} + \frac{\beta_3(\beta_1 - d - v_1)}{\beta_1} - (d + \mu) > 0,$$

**APPENDIX-B1**

Using the model  $(A_6 \dots \dots \dots A_9)$ , The general jacobian matrix can be written as follows;

$$M = \begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{matrix}$$

where

$$m_{11} = \frac{\beta_1(N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_H}{N} - d - v_1$$

$$m_{12} = \frac{\beta_1 Y_T}{N} - \frac{\beta_3 Y_T}{N}, m_{13} = \frac{-\beta_1 Y_T}{N}$$

$$m_{14} = \frac{\beta_1 Y_T}{N} - \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a)}{N^2} + \frac{\beta_3 Y_T Y_H}{N^2}$$

$$m_{21} = \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_H}{N}, m_{22} = \frac{\beta_2(N - Y_T - Y_H - Y_a - Z)}{N} - \frac{\beta_2 Y_H}{N} + \frac{\beta_3 Y_T}{N} - (d + \mu)$$

$$m_{23} = \frac{-\beta_2 Y_H}{N}$$

$$m_{24} = \frac{\beta_2 Y_H}{N} - \frac{\beta_2 Y_H (N - Y_T - Y_H - Y_a)}{N^2} - \frac{\beta_3 Y_T Y_H}{N^2}$$

**PROOF OF THEOREM-1 stability of**  
 $E_0(0,0,0, \frac{A}{d})$

The jacobian matrix for  $E_0$  is given by

$$M(E_0) = \begin{pmatrix} \beta_1 - d - \nu_1 - \nu_2 & 0 & 0 & 0 \\ 0 & \beta_2 - d - \mu & 0 & 0 \\ 0 & \mu & -(d + \alpha) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

the eigen values are  $\beta_1 - d - \nu_1, \beta_2 - d - \mu, -(d + \alpha), -(d + \theta), -d$

since  $\beta_1 > d + \nu_1, \beta_2 > d + \mu$  so

the equilibrium  $E_0$  is unstable.

**Stability for**  $E_1(\bar{Y}_T, 0, 0, \bar{N})$

The jacobian matrix for  $E_1$  is given as

$$M(E_1) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & 0 & 0 \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

where  $b_{11} = (\beta_1 - d - \nu_1), b_{12} = -(\beta_1 + \beta_3)(\beta_1 - d - \nu_1), b_{13} = -(\beta_1 - d - \nu_1)$

$$b_{14} = \frac{(\beta_1 - d - \nu_1)}{\beta_1} d^2$$

$$b_{22} = \frac{\beta_2}{\beta_1} d + \frac{\beta_3}{\beta_1} (\beta_1 - d - \nu_1) - (d + \mu)$$

Since eigen value are,  $b_{11}, b_{22}, -(\alpha + d), -(\theta + d), -d$

Since  $b_{22} > 0$  so  $E_1$  is unstable

**Stability of**  $E_2(0, \hat{Y}_H, \hat{Y}_a, \hat{N})$ ,

The jacobian for  $E_2$  is

$$M(E_2) = \begin{pmatrix} \hat{a}_1 & 0 & 0 & 0 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 \\ 0 & \mu & -(\alpha + d) & 0 \\ 0 & 0 & -\alpha & -d \end{pmatrix}$$

Where

$$\hat{a}_1 = (\mu + d)\beta - \frac{\beta_1(\beta + d)(\beta_2 - \mu - d)}{\alpha + d + \mu} - \beta_3(d + \nu_1), \hat{b}_1 = \frac{\beta_1 - \beta_2(\alpha + d)(\beta_2 - d - \mu)}{\beta_2 \alpha + d + \mu}$$

$$\hat{b}_2 = -\frac{\beta_2(\alpha + d) + (d^2 + \alpha\mu + \mu d + \alpha d)}{\alpha + d + \mu},$$

$$b_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_3 = \frac{-(\beta_2 - d - \mu)}{\alpha + d + \mu}$$

$$\hat{b}_5 = \frac{(\alpha + d)(\beta_2 - d - \mu)^2}{(\alpha + d + \mu)}$$

One of the eigen values is  $a_1$  which is  $>0$  i.e.  $E_2$  is unstable

**APPENDIX-C1 Local stability of Equilibrium**

$E_3(Y_T^*, Y_H^*, Y_a^*, N^*)$  By Lyapunov method,

The model  $(A_6 - A_9)$  can be linearised by making of following assumptions

$$Y_T = Y_T^* + y_T, Y_H = Y_H^* + y_H, Y_a = Y_a^* + y_a, N = N^* + n$$

we get

$$\frac{dY_T}{dt} = m_{11}y_T + m_{12}y_H + m_{13}y_a + m_{14}n$$

$$\frac{dY_H}{dt} = m_{21}y_T + m_{22}y_H + m_{23}y_a + m_{24}n$$

$$\frac{dy_a}{dt} = \mu y_H - (d + \alpha)y_a$$

$$\frac{dn}{dt} = -\alpha Y_a - dn$$

Where we use following Lyapunov function ;

$$V = \frac{c_1 y_T^2}{2} + \frac{c_2 y_H^2}{2} + \frac{c_3 y_a^2}{2} + \frac{c_4 n^2}{2}$$

Where  $c_1, c_2, c_3, c_4$  are to be determined.

So

$$\dot{V} = c_1 y_T \frac{dy_T}{dt} + c_2 y_H \frac{dy_H}{dt} + c_3 y_a \frac{dy_a}{dt} + c_4 n \frac{dn}{dt}$$

$$= c_1 y_T (m_1 y_T + m_2 y_H + m_3 y_a + m_4 n) + c_2 y_H (m_2 y_T + m_2 y_H + m_2 y_a + m_2 n)$$

$$+ c_3 y_a (\mu y_H - (d + \alpha) y_a) + c_4 n (-\alpha y_a - dn)$$

$$= \left[ \frac{1}{3} c_1 m_1 y_T^2 + y_T y_H (c_1 m_2) + \frac{1}{3} c_1 m_2 y_H^2 \right] + \left[ \frac{1}{4} c_1 m_1 y_T^2 + y_T y_a (c_1 m_3) - \frac{1}{4} c_1 (d + \alpha) y_a^2 \right]$$

$$+ \left[ \frac{1}{3} c_1 m_1 y_T^2 + y_T n (c_1 m_4) - \frac{1}{3} c_1 d n^2 \right]$$

$$+ \left[ \frac{1}{3} c_2 m_2 y_H^2 + y_H y_a (c_2 m_3 + \mu) - \frac{1}{4} c_2 (d + \alpha) y_a^2 \right]$$

$$+ \left[ \frac{1}{3} c_2 m_2 y_H^2 + y_H n (c_2 m_4) - \frac{1}{3} c_2 d n^2 \right]$$

$$+ \left[ \frac{1}{3} c_3 (d + \alpha) y_a^2 + y_a n (c_3) - \frac{1}{3} c_3 d n^2 \right]$$

for the equilibrium  $(y_T^*, y_H^*, y_a^*, n^*)$

$$m_1 = \frac{-\beta_1 y_T^*}{N^*}, m_2 = (-\beta_1 - \beta_2) \frac{y_T^*}{N^*}, m_3 = \frac{-\beta_1 y_a^*}{N^*}, m_4 = (\beta_1 - d - v_1) \frac{y_T^*}{N^*}$$

$$m_2 = (-\beta_1 + \beta_2) \frac{y_H^*}{N^*}, m_3 = m_3 = \frac{\beta_2 y_a^*}{N^*}, m_4 = (\beta_2 - d - \mu) \frac{y_H^*}{N^*}$$

since all the coefficients of quadratic terms are negative, it will be negative definite provided the following conditions are satisfied

(i)

$$\left[ c_1 (\beta_1 + \beta_2) \frac{y_T^*}{N^*} + c_2 (\beta_2 - \beta_1) \frac{y_H^*}{N^*} \right]^2 < \frac{4c_1 c_2 \beta_1 \beta_2 y_T^* y_H^*}{9N^{*2}}$$

$$c_1 \beta_1 y_T^* < \frac{4c_3 N^* (d + \alpha)}{9}$$

(ii)

$$c_1 y_T^* (\beta_1 - d - v_1)^2 < \frac{4c_4 \beta_1 N^* d}{9}$$

(iii)

(iv)

$$\left[ \frac{c_2 \beta_2 y_H^*}{N^*} - \mu c_3 \right]^2 < \frac{4c_2 c_3 (d + \alpha) \beta_2 y_H^*}{9N^{*2}}$$

$$c_2 y_H^* (\beta_2 - d - \mu)^2 < \frac{4c_4 d \beta_2 N^*}{9}$$

(v)

$$c_4 \alpha^2 < \frac{4c_3 d (d + \alpha)}{9}$$

(vi)

**APPENDIX-D1  
(NON LINEAR STABILITY)  
PROOF OF THE LEEMA**

From model  $(A_6 \dots A_9)$  we can obtain as follows;

$$\frac{dN}{dt} = A - dN - \alpha Y_a \leq A - dN \Rightarrow N \leq \frac{A}{d}$$

$$\frac{dY_T}{dt} = \frac{\beta_1 Y_T (N - Y_T - Y_H - Y_a)}{N} - \frac{\beta_3 Y_T Y_H}{N} - (d + v_1) Y_T$$

$$\leq (\beta_1 - d - v_1) Y_T - \frac{\beta_1 Y_T^2}{N}$$

$$\Rightarrow Y_T \leq \frac{A(\beta_1 - d - v_1)}{d\beta_1}$$

$$Y_H \leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$$

similarly and

$$\frac{dY_a}{dt} = \mu Y_H - dY_a - \alpha Y_a$$

$$\Rightarrow Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d (d + \alpha)}$$

so the region of attraction is

$$0 \leq Y_T \leq \frac{A(\beta_1 - d - v_1 - v_2)}{d\beta_1}$$

$$0 \leq Y_H \leq \frac{(\beta_2 - d - \mu)A}{\beta_2 d}$$

$$0 \leq Y_a \leq \frac{(\beta_2 - d - \mu)A\mu}{\beta_2 d (d + \alpha)}$$

$$0 \leq N \leq \frac{A}{d}$$

**PROOF OF THE THEOREM**

In this case we use the following Lyapunov function;

$$W = \left( Y_T - Y_T^* - Y_T^* \log \frac{Y_T}{Y_T^*} \right) + m_1 \left( Y_H - Y_H^* - Y_H^* \log \frac{Y_H}{Y_H^*} \right) + \frac{m_2}{2} (Y_a - Y_a^*)^2 + \frac{m_3}{2} (N - N^*)^2$$

on differentiating and substituting the values from model we get:

$$\begin{aligned} & \left[ \frac{-\beta_1}{3N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_H - Y_H^*) \left\{ \frac{-\beta_1 - \beta_2 - \beta_3 m_1 + \beta_3 m_1}{N^*} \right\} + \frac{1}{3} (Y_H - Y_H^*) \left( \frac{-\beta_2 m_2}{N^*} \right) \right] \\ & + \left[ \frac{-\beta_1}{3N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (Y_a - Y_a^*) \left\{ \frac{-\beta_1 m_3}{N^*} - \frac{1}{4} m_3 (d + \alpha) \right\} + (Y_a - Y_a^*)^2 \right] \\ & + \left[ \frac{-\beta_1}{3N^*} (Y_T - Y_T^*)^2 + (Y_T - Y_T^*) (N - N^*) \left\{ \frac{\beta_1 Y_T + \beta_2 Y_H + (\beta_1 + \beta_2) Y_a}{NN^*} \right\} - \frac{1}{3} d m_3 (N - N^*)^2 \right] \\ & + \left[ \frac{1}{3} (Y_H - Y_H^*) \left( \frac{-\beta_2 m_2}{N^*} \right) + (Y_H - Y_H^*) (Y_a - Y_a^*) \left\{ \frac{-\beta_2 m_2}{N^*} + \mu m_3 \right\} - \frac{1}{3} m_2 (d + \alpha) (Y_a - Y_a^*)^2 \right] \\ & + \left[ \frac{1}{3} (Y_a - Y_a^*) \left( \frac{-\beta_3 m_1}{N^*} \right) + (Y_a - Y_a^*) (N - N^*) \left\{ \frac{\beta_3 m_1 Y_T + (\beta_1 + \beta_2) Y_H - \beta_3 m_1 Y_T}{NN^*} \right\} - \frac{1}{3} d m_3 (N - N^*)^2 \right] \\ & + \left[ -\frac{1}{3} m_3 (d + \alpha) (Y_a - Y_a^*)^2 + (Y_a - Y_a^*) (N - N^*) (-m_3 \alpha) - \frac{1}{3} d m_3 (N - N^*)^2 \right] \end{aligned}$$

CONS

$$\frac{Y_T}{N}, \frac{Y_H}{N}, \frac{Y_a}{N} = 1$$

IDE

$W^0$  is negative definite if the following conditions are satisfied ;

$$(i) (\beta_1 + \beta_3 + \beta_2 m_1 - \beta_3 m_1)^2 < \frac{4\beta_1\beta_2 m_1}{9}$$

$$(ii) \beta_1 < \frac{4m_2(d + \alpha)N^*}{9}$$

$$(iii) (3\beta_1 + \beta_3)^2 < \frac{4dm_3\beta_1 N^*}{9}$$

$$(iv) \left( \frac{-\beta_2 m_1}{N^*} + \mu m_2 \right)^2 < \frac{4m_1 m_2 (d + \alpha) \beta_2}{9 N^*}$$

(v)

$$(\beta_2 m_1 + \beta_2 m_2 + \beta_3 m_1)^2 < \frac{4\beta_2 m_1 m_3 d N^*}{9}$$

$$(iv) m_3 \alpha^2 < \frac{4m_2 d (d + \alpha)}{9}$$

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